Senior Thesis Plan

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The Dirichlet class number formula relates the behavior of the zeta function ζ_K of a number field K (which is an analytically defined function) around 1 to certain quantities that characterize the algebraic structure of K:

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{w_K\sqrt{|d_K|}},$$

where r_1 and r_2 represent the number of real and complex embeddings of K, h_K is the class number, R_K is the regulator of K, w_K represents the number of units in K and d_K is the discriminant of K.

Analogous formulae exist for various other objects; for the case of abelian varieties the conjectured formula is due to Birch and Swinnerton-Dyer. As before, it relates certain algebraic quantities that characterize the abelian variety to an analytically defined L-function.

Let K be a global field, A an abelian variety of dimension d defined over K, $A^\vee$ the dual abelian variety. Let r be the algebraic rank of A, meaning that there exist independent points $a_1, \ldots, a_r \in A$. Let $a'_1, \ldots, a'_r$ be their corresponding independent points on $A^\vee$. Using the canonical height pairing we defined the regulator to be $R = \det\langle a_i, a'_j\rangle$.

The w term in the class number formula is replaced by $T = [A : \sum \mathbf{Z}a_i][A^\vee : \sum \mathbf{Z}a'_i]$, a measure of torsion. The corresponding quantity for the class number h_K is the cardinality of the Shafarevich-Tate group $Ш(A,K)$ defined to be the kernel of the natural inclusion

$$H^1(K, A(\bar{K})) \to \bigoplus_v H^1(K_v, A(\bar{K}_v)).$$

In the case when K is a number field, the Birch and Swinnerton-Dyer conjecture appears in the literature stated in two ways:

$$\frac{(L^{an})^{(r)}(A,1)}{r!} = \frac{\det\langle a_i, a'_j\rangle |Ш(A,K)|}{[A : \sum \mathbf{Z}a_i][A^\vee : \sum \mathbf{Z}a'_i]}$$
$$\frac{(L^{al})^{(r)}(A,1)}{r!} = \frac{\det\langle a_i, a'_j\rangle |Ш(A,K)| \prod_v c_v}{\sqrt{d_K}^d [A : \sum \mathbf{Z}a_i][A^\vee : \sum \mathbf{Z}a'_i]},$$

where c_v are the Tamagawa number associated with a place v (if v is a finite place then c_v is defined to be the cardinality of the component group of the special fiber over v in the Néron model of A over $\mathcal{O}_K$) and L^{an}, L^{al} are the L-functions of A. (For this conjecture to make sense, it must be conjectured that the analytic rank of A is r, that the L-functions have analytic continuations to a neighborhood of 1 and that the Shafarevich-Tate group is finite.)

The first part of the thesis proves the equivalence of these two statements. The strategy is to relate the Tamagawa numbers c_v to $(L^{al})^{(r)}(A, 1)/(L^{an})^{(r)}(A, 1)$. To do this we pass to the Néron model $\mathcal{A}$ of A over $\mathcal{O}_K$. The connected component of the identity of the special fiber $\mathcal{A}_v$ of $\mathcal{A}$ over v has (by Chevalley's theorem) an abelian part B, a toric part $\mathbf{T}$ and a unipotent (additive) part N. The equivalence of the two statements will come down to relating the number of points of each of these parts over the finite field k_v to the characteristic polynomial of the arithmetic Frobenius σ_v acting on them.

The conjecture has been proven in its weak form (that the analytic rank is r) for elliptic curves of rank 0 and 1. The general form has been computationally checked for many such elliptic curves using a variant of Kolyvagin's theory of Euler systems. However, the only proven facts about the conjecture in general are certain consistency properties.

The thesis will try to cover the following:

1. The conjecture is invariant under restriction of scalars.

2. The conjecture is invariant under isogenies.

3. Possible computational aspects of the conjecture.

The isogeny invariance of the conjecture was proven by Tate, who generalized the corresponding proof for elliptic curves, due to Cassels. It uses the global Euler-Poincaré characteristic formula and Tate-Poitou duality. Once these technical details are understood, deducing the isogeny invariance follows from the existence of the Cassels pairing between $Ш(A, K)$ and $Ш(A^\vee, K)$.

Possible computational topics include examples where each of the quantities that appear in the conjecture changes under isogeny (while the overall formula is invariant) as well as a discussion of the finiteness of $Ш(A, K)$.

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