Morse Theory and Hyperplane Sections of Algebraic Varieties

Andrei Jorza

December 10, 2003

Abstract

The purpose of this paper is to discuss the relation between Morse theory and the topology of algebraic varieties using hyperplane sections. The first part of the paper introduces general Morse theory needed to analyze the general topology of affine algebraic varieties. The second part of the paper discusses the Lefschetz theory of complex projective varieties. The last part of the paper gives an overview of stratified Morse theory for noncompact affine algebraic varieties and proves a theorem about the Euler characteristic.

1 Introduction

Let $M \subset \mathbb{R}^m$ be an n-dimensional manifold. Denote T_pM the tangent space to M at a point $p \in M$. Let $TM = \bigcup_{p\in M} T_pM$ be the tangent bundle.

For a manifold X and a function $f : M \longrightarrow X$ we have the derivative $df_p = f_* : T_pM \longrightarrow T_{f(p)}X$. A point $p \in M$ is called **critical** if df_p does not have full rank. If p is a critical point then $f(p)$ is called a **critical** value. In a local neighborhood of p we have coordinates $x_1, \ldots, x_n : \mathbb{R}^n \longrightarrow M$ which define a diffeomorfism between an open ball in $\mathbb{R}^n$ and the neighborhood of p. In that case $f_* = (\frac{\partial f}{\partial x_i})$.

In a neighborhood of p we can define the **Hessian** of a real function f to be the matrix representing the second derivative of f,

$$f_{**} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right).$$

Note that $f_{**} : T_pM \times T_pM \longrightarrow \mathbb{R}$ is a symmetric bilinear form.

A critical point p is called **nondegenerate** if the Hessian of f at p has rank n. The **index** of p in that case represents the number of negative eigenvalues of the Hessian.

2 Morse Functions

If all the critical points of a real function f are nondegenerate then f is called a **Morse function**. The main purpose of this section is to prove the existence of Morse functions of the form $\| x - p \|^2$ for a "nonfocal" point p.

Let N be the normal bundle of M, i.e., $N = \{(p, v) | p \in M, v \in \mathbb{R}^m, v \perp T_pM\}$.

Define $e : N \longrightarrow \mathbb{R}^m$ by $e(p, v) = p + v$, where $p \in M \subset \mathbb{R}^m$.

Definition 1 *A point $x \in \mathbb{R}^m$ is called* **focal** *if it is a critical value of the function e. If $e(p, v) = x$ then we say that x if* **focal** *p.*

Lemma 1 *There exist nonfocal points.*

Proof: Clearly e is a smooth function between manifolds so we may use Sard's theorem to prove that the set of critical value of e, i.e., the set of focal points, has measure zero. Therefore there exists a point $x \in \mathbb{R}^m$ which is not focal. ∎

We will look for Morse functions among the functions representing the square of distance to a point.

First we need a lemma in linear algebra.

Lemma 2 *Let $e_1, \ldots, e_n$ be the columns of a matrix A and let $v_1, \ldots, v_n$ be a basis of the space. Then $rank(A) = rank(e_i v_j)$.*

Proof: Note that any linear dependence of the e_i's corresponds to a linear dependence of the columns of $(e_i v_j)$ and vice-versa, because the v_j's form a basis. The result follows from the fact that rank is the maximal linear independence. ∎

Let $x : M \longrightarrow \mathbb{R}^m$ be the natural inclusion.

Theorem 3 *For nonfocal $p \in \mathbb{R}^m$, the function $f(x) = \| x - p \|^2$ is Morse.*

Proof:

We have $f(x(q)) = x(q)^2 - 2x(q)p + p^2$, where the dot product is inherited from $\mathbb{R}^m$. Then for coordinate functions $u_1, \dots, u_n$ in a neighborhood of q we have $\frac{\partial f}{\partial u_i} = 2\frac{\partial x}{\partial u_i}(x(q) - p) = 2x_* \frac{\partial}{\partial u_i}(x(q) - p)$. Since $\frac{\partial}{\partial u_i}$ generate $T_q M$ it means that $x_* \frac{\partial}{\partial u_i}$ generate $T_{x(q)} M \subset \mathbb{R}^m$. Therefore q is a critical point if $x(q) - p \perp x_* \frac{\partial}{\partial u_i}$ for all i, so if $x(q) - p \perp T_{x(q)} M$.

Let $l = \| p - x(q) \|$ and then $p = x(q) + lv$ for some unit vector v. For simplicity, I will write q instead of $x(q)$. Then the Hessian of f at q is

$$\begin{aligned} f_{**} &= \left(\frac{\partial^2 f}{\partial u_i \partial u_j} \right) = \left(\frac{\partial}{\partial u_i} 2 \frac{\partial f}{\partial u_j}(x(q) - p) \right) \\ &= 2 \left(\frac{\partial^2 x}{\partial u_i \partial u_j}(q - p) + \frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_j} \right) = 2(g_{ij} - lvf_{ij}) = 2(I_n - lF) = 2l(\frac{1}{l} I_n - F) \end{aligned}$$

where $g_{ij} = \frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_j}$, $f_{ij} = \frac{\partial^2 x}{\partial u_i \partial u_j}$ and $F = (vf_{ij})$. By the spectral theorem there is a choice of coordinate functions u_i, so that $(g_{ij}) = I_n$. This formula gives the following proposition:

Proposition 4 *f is degenerate at q if and only if $1/l$ is an eigenvalue of F.*

Proposition 5 *If $F = (vf_{ij})$ has nonzero eigenvalues $e_1, \dots$ then the focal points at q (which all lie on the line determined by v) are $q + e_i^{-1} v$, for all i.*

Proof: Choose $w_1, \dots, w_{m-n}$ vector fields in the perpendicular bundle so that they are orthonormal in a neighborhood of q. Then any vector w which is perpendicular to $T_q M$ ca be written as a linear combination $w = \sum \lambda_i w_i$.

Then $e(q, w) = x(q) + \sum \lambda_k w_k$ and so we have that

$$\begin{aligned} \frac{\partial e}{\partial u_i} &= \frac{\partial x}{\partial u_i} + \sum \lambda_k \frac{\partial w_k}{\partial u_i} \\ \frac{\partial e}{\partial \lambda_k} &= w_k \end{aligned}$$

Let $w = tv$. Then this is a focal point at q if and only if it is a critical point of e, i.e., if the rank of e is less than m.

Since $x_* \frac{\partial}{\partial u_i}$ are a basis of $T_{x(q)} M$ and w_k are a basis of the perpendicular space at q it means that they form a basis of the perpendicular bundle. Therefore we can apply the lemma to get that the matrix

$$\begin{pmatrix} \frac{\partial e}{\partial u_i} x_* \frac{\partial}{\partial u_j} & \frac{\partial e}{\partial u_i} w_k \\ \frac{\partial e}{\partial \lambda_k} x_* \frac{\partial}{\partial u_j} & \frac{\partial e}{\partial \lambda_k} w_j \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial u_i}\frac{\partial x}{\partial u_j} + \sum \lambda_k \frac{\partial w_k}{\partial u_i}\frac{\partial x}{\partial u_j} & \sum \lambda_j \frac{\partial w_j}{\partial u_i} w_k \\ w_i x_* \frac{\partial}{\partial u_j} & w_i w_j \end{pmatrix}$$

has rank less than m.

Now $(w_i w_j) = I_{m-n}$ and $w_i x_* \frac{\partial}{\partial u_j} = 0$ by definition. Also $\frac{\partial}{\partial u_i}(w_k x_* \frac{\partial}{\partial u_j}) = \frac{\partial w_k}{\partial u_i}\frac{\partial x}{\partial u_j} + w_k \frac{\partial^2 x}{\partial u_i \partial u_j}$ This matrix becomes

$$\begin{pmatrix} g_{ij} - \sum \lambda_k w_k f_{ij} & \sum \lambda_j \frac{\partial w_j}{\partial u_i} w_k \\ 0 & I_{m-n} \end{pmatrix}$$

Now the rank of this matrix is $m-n$ plus the rank of the matrix $(g_{ij} - tvf_{ij}) = I_n - tF = t(\frac{1}{t}I_n - F)$. Therefore the focal points are precisely the $q + e_i^{-1}v$ as needed. Note that the multiplicity of the eigenvalues determine the indexes of the focal points. ■

The theorem is now a direct consequence of the previous two propositions. Since focal points are a measure zero set we get the following corollary:

Corollary 6 *For almost all p the function f is Morse.*

3 Complex Manifolds

3.1 General Theory

A complex analytic manifold is a manifold whose ringed structure is that of complex analytic functions. Unlike for real manifolds, not all complex analytic manifolds can be embedded in some $\mathbb{C}^n$.

Let M be an n-dimensional complex analytic manifold embedded in some $\mathbb{C}^m$. For the following proposition see [Morrow-Kodaira].

Proposition 7 *A complex analytic manifold is orientable.*

Proposition 8 *For any $q \in M$, the points p which are focal at q come in symmetric pairs with respect to q.*

Proof: Let $z_1, z_2, \ldots, z_n$ be local coordinates near q (this implies that $z_i(q) = 0$) and let $w : M \longrightarrow \mathbb{C}^m$ be the embedding. Since w is analytic, its components are also analytic so they have a power series expansion. As before, $p = q + vl$.

Look at the dot product inherited from the Hermitian one on $\mathbb{C}^m$. We have $wv = C + L + Q(z_1, z_2, \ldots, z_n) + H$, where C is a constant, L are the linear terms, Q is quadratic homogeneous, and H stands for higher order terms. Now, $v \perp T_{w(q)}M$ so that means that linear terms do not survive. So $L = 0$. Transform M into a real manifold via the standard $z_j = x_j + iy_j$ so get that $wv = C_r + Q_r(x_1, y_1, \ldots, x_n, y_n) + H_r$ (by taking real parts).

Since v is fixed then $F = (v \cdot \frac{\partial^2 w}{\partial\alpha\partial\beta}) = (\frac{\partial^2 vw}{\partial\alpha\partial\beta})$, where $\alpha, \beta \in \{x_1, \ldots, y_n\}$. Since $z_i(q) = 0$ the second partial derivatives of terms in H vanish at q, and so do the constant terms. So the only terms that count are the ones in Q_r. But then the eigenvalues of the matrix $(\frac{\partial^2 Q_r}{\partial\alpha\partial\beta})$ are precisely the eigenvalues of Q_r. Since $Q(iz) = -Q(z)$ it means that there is an orthogonal transformation that changes the sign of Q_r as well. Therefore, the eigenvalues of Q_r come in pairs of opposite signs and equal multiplicities.

Since the focal points are translates of q along v by reciprocals of eigenvalues, this means that focal points are symmetrical with respect to q. ∎

3.2 Morse Theory

We now present the main theorem of Morse theory, which is proven in [Milnor]:

Theorem 9 *If M is a manifold and f is a Morse functions so that for each $x \in \mathbb{R}$ the set $f^{-1}((-\infty, x])$ is compact, then M is homotopy equivalent to a CW complex with cells of dimension k for each critical point of f of index k.*

In our case, f is the square of the distance function. We have seen that $f_{**} = 2l(\frac{1}{l}I_n - F)$ at q. Moreover, the index of q is the number of negative eigenvalues of f_{**}. But if e_i are the eigenvalues of F then the eigenvalues of f_{**} are $2l(\frac{1}{l} - e_i)$ so the number of negative eigenvalues is the number of e_i so that $l > \frac{1}{e_i}$ (of course, we need $e_i > 0$). This means that $q + e_i^{-1}v$ is between q and $p = q + lv$. So the index at q is the number of focal points between p and q along v.

Theorem 10 *If M is an n-dimensional complex analytic embedded as a closed set in some $\mathbb{C}^m$ then M has the homotopy type of a CW complex with cells of dimension at most n.*

Proof: The CW complex structure follows from the main theorem of Morse theory applied to the Morse function $f(x) = \| x - p \|^2$. Since $f(x) \geq 0$ it means that $f^{-1}((-\infty, x]) = f^{-1}([0, x])$ which is compact so the conditions of the theorem are satisfied.

The dimension of a cell is the index of a critical points, which is the number of focal points between p and q. But the focal points are symmetrical with respect to q so at most one half the total number of focal points (i.e., $2n$ since there are most $2n$ eigenvalues for the quadratic form Q_r) can be between p and q. So the cells have dimension at most n. ∎

4 Projective Algebraic Varieties

It is a direct consequence of the maximum principle that compact complex manifolds cannot be embedded in some $\mathbb{C}^m$. Therefore the theory in the previous section would not apply.

First, we need the following theorem due to Poincare and Lefschetz, whose proof can be found in [Hatcher]:

Theorem 11 *(duality theorem)*

1. *(Poincare) If M is a compact orientable manifold of dimension n then there is an isomorphism*

$$H^i(M) \xrightarrow{\sim} H_{n-i}(M)$$

2. *(Lefschetz) If M is a compact orientable manifold of dimension n with boundary ∂M then there is an isomorphism*

$$H^i(M, \partial M) \xrightarrow{\sim} H_{n-i}(M)$$

Let M be a complex n-dimensional projective algebraic variety sitting in some $\mathbb{C}P^m$. Let π be a hyperplane in $\mathbb{C}P^m$ containing all the singularities of M if there are any.

Theorem 12 *(Lefschetz)*

1. *If $i < n-1$ then $H_i(M \cap \pi) \cong H_i(M)$.*
2. *For $i = n-1$, the natural inclusion $H_i(M) \longrightarrow H_i(M \cap \pi)$ is surjective.*

Proof:
Consider the following long exact sequence of the pair $(M, M \cap \pi)$ long exact sequence

$$\ldots \longrightarrow H_{i+1}(M, M \cap \pi) \longrightarrow H_i(M \cap \pi) \longrightarrow H_i(M) \longrightarrow H_i(M, M \cap \pi) \longrightarrow \ldots$$

Lemma 13

$$H_i(M, M \cap \pi) \cong H^{2n-i}(M \setminus (M \cap \pi)).$$

Proof: Apply the Lefschetz duality theorem for $M \setminus (M \cap \pi)$ which is an orientable manifold with boundary π. ∎

Now, identifying $\mathbb{C}P^m \setminus \pi = \mathbb{C}^m$, we get that $M \setminus (M \cap \pi)$ is an affine algebraic variety. Moreover, since π contains all the singularities of M it means that it is a nonsingular affine algebraic veriety, so we may apply the theory of the previous section. Therefore $M \setminus (M \cap \pi)$ has the homotopy type of a CW complex with cells of dimension $\leq n$.

This means that all cohomology groups of order $> n$ are zero so for $i < n$ we have $H_i(M, M \cap \pi) \cong H^{2n-i}(M \setminus (M \cap \pi)) = 0$. Therefore for $i < n-1$ we get a piece of the long exact sequence:

$$0 \longrightarrow H_i(M \cap \pi) \longrightarrow H_i(M) \longrightarrow 0,$$

while for $i = n-1$ the piece is

$$H_i(M \cap \pi) \longrightarrow H_i(M) \longrightarrow 0.$$

The results follow from this. ∎

5 Computational Topology

The purpose of this section is to give a general method to compute the Betti numbers of projective algebraic varieties.

5.1 Algebraic varieties defined by polynomials of the same degree

Let M be a complex projective algebraic variety defined by a nonsingular system of homogeneous polynomials $P_1, \ldots, P_s$, each of degree d. So $P_j(z_0, \ldots, z_m) = \sum a^j_{i_0,\ldots,i_m} z_0^{i_0} \cdots z_m^{i_m}$ with $\sum i_j = d$. This variety is the zero set of a smooth function, so it has codimension s (the same as the codimension of $\{0\}$ in $\mathbb{R}^s$) so it has dimension $m - s$.

Definition 2 *The Veronese embedding is $\psi : \mathbb{C}P^m \longrightarrow \mathbb{C}P^{\binom{m+d}{d}-1}$ taking the k-th coordinate of ψ to the k-th term $z_0^{i_0} \cdots z_m^{i_m}$ in lexicographic order.*

The following proposition is obvious:

Proposition 14 *The Veronese embedding, ψ, has no critical points and is a local diffeomorphism between $\mathbb{C}P^m$ and $\psi(\mathbb{C}P^m)$. This implies that the Betti numbers are invariant under ψ.*

Let ${l^j}_1, {l^j}_2, \ldots, {l^j}_t$ be the indices of the $(i_0^j, {i^j}_1, {i^j}_2, \ldots, {i^j}_m)$ in the lexicographic ordering, for which $a^j_{i_0,i_1,i_2,\ldots,i_m}$ are nonzero. Let h_k be homogeneous coordinates for $\mathbb{C}P^{\binom{m+d}{d}} - 1$.

The following proposition follows from definition.

Proposition 15 $\psi(M) = \psi(\mathbb{C}P^m) \cap \pi_1 \cap \ldots \cap \pi_s$, *where* $\pi_j = \{h_{l_1^j} + \cdots + h_{l_t^j} = 0\}$.

We are now ready to state the main theorem for the Betti numbers.

Theorem 16 *For i so that $|i - m| > s + 1$ we have that $b_i(M) = 1$ if i is even and 0 if i is odd.*

Proof: Let $i < m - s - 1$. Then $\psi(\mathbb{C}P^m)$ is an algebraic variety which has no singularities so we get that if $i < m - 1$ then

$$H_i(\psi(\mathbb{C}P^m)) \cong H_i(\psi(\mathbb{C}P^m) \cap \pi_1).$$

Also, $\psi(\mathbb{C}P^m) \cap \pi_1$ is also an algebraic variety with no singularities (the Jacobian has full rank) so we may apply the theorem again. Repeat this step until we get to $H_i(\psi(M))$:

$$\begin{aligned} H_i(\psi(\mathbb{C}P^m)) &\cong H_i(\psi(\mathbb{C}P^m) \cap \pi_1) \cong H_i(\psi(\mathbb{C}P^m) \cap \pi_1 \cap \pi_2) \cong \ldots \\ &\cong H_i(\psi(\mathbb{C}P^m) \cap \bigcap \pi_j) = H_i(\psi(\mathbb{C}P^m) \cap \pi) = H_i(\psi(M)). \end{aligned}$$

(It is implicit that the dimension of each $\psi(\mathbb{C}P^m) \cap \pi_1 \cap \ldots \cap \pi_j$ is $\geq m - j$ so the fact that $i < m - s - 1$ is enough for our purposes.)

Therefore for $i < m - s - 1$ we have that $b_i(M) = b_i(\mathbb{C}P^m)$ and for $i > m + s + 1$ the same relation follows from Poincare duality. ∎

5.2 Algebraic varieties defined by polynomials of different degrees

I will only treat the case of a complex projective algebraic variety M defined by two polynomials, one of degree a and the other of degree b.

As before, consider the Veronese embeddings $\psi_a : \mathbb{C}P^m \longrightarrow \mathbb{C}P^A, \psi_b : \mathbb{C}P^m \longrightarrow \mathbb{C}P^B$, where $A = \binom{m+a}{a} - 1, B = \binom{m+b}{b} - 1$.

Definition 3 *The Segre embedding is the embedding* $\Phi : \mathbb{C}P^A \times \mathbb{C}P^B \longrightarrow \mathbb{C}P^{AB+A+B}$ *which takes the homogeneous coordinate* (u_i, v_j) *to* $(u_i v_j)$. *Denote the homogeneous polynomials of this projective space by* h_{ij} *in the obvious way.*

This function is well defined since all the $u_i v_j$ have degree 2. Let $\psi = \Phi(\psi_1, \psi_2)$.

Proposition 17 *WLOG assume that* $u_0, \ldots, u_p$ *are the homogeneous coordinates of* $\mathbb{C}P^A$ *that represent the monomials in the first polynomial describing* M. *Similarly define* $v_0, \ldots, v_q$. *Then*

$$\psi(M) = \psi(\mathbb{C}P^m) \cap \pi_0^1 \cap \ldots \cap \pi_B^1 \cap \pi_0^2 \cap \ldots \cap \pi_A^2,$$

where $\pi_j^1 = \{\sum_{k=0}^p h_{kj} = 0\}, \pi_j^2 = \{\sum_{k=0}^q h_{jk} = 0\}$.

Proof: This intersection is determined on each of the two Veronese projective spaces by the equations $\sum u_k v_j = 0$ for all j. But not all v_j are 0 so this means that $\sum u_k = 0$ and this is equivalent to the first polynomial relation defining M. Similarly for the second defining polynomial. Therefore the conclusion follows. ■

Remark 4 *The method presented in the previous section allows us to calculate the Betti numbers for* i *so that* $|i - m| > 3$.

6 Application to Homotopy Theory

The following lemma is a direct application of the Tubular Neighborhood Lemma ([Guillemin-Pollack] pp. 76):

Lemma 18 *For a smooth submanifold* $M \subset N$, *there exists a neighborhood* $V \subset N$ *that is diffeomorphic to* $[-1, 1] \times M$, *with* M *identified with* $0 \times M$. *Also,* V *retracts to* M.

The main application is the following theorem:

Theorem 19 *In the context of Lefschetz's theorem for hyperplane sections, if* M *is nonsingular then we have*

$$\pi_k(M, M \setminus \pi) = 0,$$

for all $k < n$.

Proof: For a nonfocal point p, define $g(x) = 0, x \in M \cap \pi$ and $g(x) = \| x - p \|^{-2}$ for $x \notin M \cap \pi$. Since $\| x - p \|^2$ is Morse and each critical point has index $\leq n$ it means that all critical points of g have index at least n. Moreover, for $\varepsilon > 0$ we have that $g|_M$ is nondegenerate because $\| x - p \|^2$ is.

So M has the homotopy type of $g^{-1}([0, \varepsilon])$ with finitely many cells of dimension $\geq n$ attached. For ε small enough so that $g^{-1}([0, \varepsilon]) \subset V$ (as in the lemma applied to $M \cap \pi$). By Hurewicz's theorem, since the homology groups are zero in dimension $k < n$ for the attached cells, the homotopy group π_k will be determined by the $g^{-1}([0, \varepsilon]) \subset V$.

Now any map from $([0,1]^k, \partial[0,1]^k)$ to $(g^{-1}([0, \varepsilon]), M \cap \pi) \subset (V, M \cap \pi)$ can be deformed to a map going to $M \cap \pi$ (using the retraction from V to $M \cap \pi$), which proves that the homotopy groups are 0 in dimension less than n. ■

7 Noncompact Complex Manifolds

Let M be an affine algebraic variety of dimension n embedded as a closed set in some $\mathbb{C}^m$. Then M is no longer compact so Poincare duality does not hold. We know from theorem 8 that M has the structure of a CW complex with cells of dimension at most n. This means that $H_i(M) = 0$ for all $i > n$. Moreover, Poincare duality for noncompact manifolds ([Hatcher] pp. 245) gives the following result:

Proposition 20 *$H_c^i(M) = 0$ for all $i < n$, where H_c^* represents cohomology with compact support.*

Using the method of hyperplane sections, one can get results about the Euler characteristic of the manifold. For algebraic varieties the homology groups are finitely generated so the following notion is well-defined:

Definition 5 *The Euler characteristic of an n-dimensional manifold M is*

$$\chi(M) = \sum_{k=0}^{n} (-1)^k b_k(M).$$

7.1 An extension of Morse theory

In this section, I will follow the arguments in [Kaveh], citing prerequisites without proof. This section should be seen as an overview to theory, rather than an exposition of it.

The main problem with applying results from classical Morse theory to noncompact manifolds is that for a general Morse function f, the set $f^{-1}((-\infty, x])$ might not be compact. The solution is to extend the results presented in the previous sections of this paper.

7.1.1 Whitney Stratifications

An analogue to the theory of (non-)focal points is the following notion to Whitney:

Definition 6 *Let M be a manifold. The Whitney stratification is a partially ordered collection of locally closed submanifolds $\mathcal{S}_\alpha$ with the following property:*

Let $\mathcal{S}_1 < \mathcal{S}_2$ and sequences $(x_i) \subset \mathcal{S}_1, (y_i) \subset \mathcal{S}_1$ that converge to some $x \in \mathcal{S}_1$. Then if the lines $\overline{x_i y_i}$ (in a local neighborhood) converge to a line l and if $T_{y_i}\mathcal{S}_1$ converge to some plane π then $l \subset \pi$ and $T_x\mathcal{S}_1 \subset \pi$.

Theorem 21 *Any complex algebraic variety admits a Whitney stratification with a finite number of strata.*

The proof of this theorem ([Kaloshin]) exhibits Whitney stratifications as related to regular values for functions involving angles between tangent spaces. The result then follows from Sard's theorem.

7.1.2 Generalized Morse Theory

This generalization is due to Palais and Smale in [Palais-Smale].

Theorem 22 *Let M be an n-dimensional complex manifold embedded as a closed set in $\mathbb{C}^m$ and f is a Morse function such that for any set S on which $|f|$ is bounded but $\| \nabla f \|$ is not bounded away from 0, there is a critical point of f in $\overline{S}$. Then the main theorem of Morse theory holds for M.*

Theorem 23 *Let M be an n-dimensional algebraic variety embedded as a closed subset of $\mathbb{R}^m$, with finite Whitney stratification for the closure of M in $\mathbb{R}P^{m-1}$ so that the strata that contain the north pole are transverse to the hyperplane $f^{-1}(0)$, where f is a linear functional. Then f satisfies the condition of the previous theorem.*

Proof: ([Kaveh]) Assume that the condition is not satisfied. Then there is a sequence $(x_n) \subset M$ so that (x_n) is unbounded (if it is bounded then the problem reduces to compact manifold in which the problem is easy) and $\| \nabla f(x_n) \| \longrightarrow 0$. But $Gr(n, m)$ is compact so $T_{x_n}M$ has a converging subsequence, which converges to some π.

We know that $\| \nabla f(x_n) \| \longrightarrow 0$ so the angle between ∇f and $T_{x_i}M$ goes to 0 so it must be (by passing to the limit) that the angle between π and ∇f is 0. But f is linear so this means that π lies in the zero set of f. Therefore if $x_n \longrightarrow x$ in $\overline{M}$ and $\mathcal{S}$ is the stratum containing x then $T_x\mathcal{S} \subset \pi \subset f^{-1}(0)$. But then this contradicts that $f^{-1}(0)$ is transverse to $\mathcal{S}$ which contains the north pole (since (x_n) is unbounded). ∎

Remark 7 *Almost every linear functional is Morse and almost every Morse linear functional has the property that it is transverse to a finite number of strata containing the pole in the projective space.*

7.1.3 Euler characteristic of noncompact manifolds

We may now state the theorem for the Euler characteristic of hyperplane sections.

Theorem 24 *Let M be an algebraic variety of dimension n, embedded as a closed set in some $\mathbb{C}^m$. Then for almost all hyperplanes π we have*

$$\chi(M) = \chi(M \cap \pi) + (-1)^n c(\pi)$$

(where $c(\pi)$ represents the number of critical points of the linear function defining π, when restricted to M).

Proof: Let f be the linear function defining π. From the main theorem of generalized Morse theory, we get that

$$\chi(M) = \chi(M \cap \pi) + \sum (-1)^i c_j$$

where i is the index of the critical point c_j. But we are working over complex numbers so the index is equal to the complex dimension of the manifold ($-1 = i^2$ and $(x+iy)^2 = x^2 - y^2 + 2ixy$ so $f(z) = \sum z_i^2$ locally so $Ref(z) = \sum x_i^2 - \sum y_i^2$ so the index equals the complex dimension). Therefore all $i = m$ and the conclusion follows. ■

References

[Guillemin-Pollack] Guillemin, V., Pollack, A., *Differential Topology*, NJ: Englewood Cliffs, 1974.

[Hatcher] Hatcher, Allen, *Algebraic Topology,* Cambridge University Press, 2001.

[Kaloshin] Kaloshin, V., *A geometric proof of the existence of Whitney stratifications*, http://arxiv.org/abs/math.AG/0010144 (accessed 12/9/2003).

[Kaveh] Kaveh, K., *Morse theory and the Euler characteristic of sections of spherical varieties*, http://arxiv.org/abs/math.AG/0112064 (accessed 12/9/2003).

[Milnor] Milnor, J., *Morse Theory, based on lecture notes by M. Spivak and R. Wells,* Princeton: Princeton University Press, 1973.

[Morrow-Kodaira] Morrow, J., Kodaira, K., *Complex Manifolds*, Holt, Rinehart and Winston Inc., 1971.

[Palais-Smale] Palais, R.S., Smale, S., *A Generalized Morse Theory,* in *Bulletin of the Amer. Math. Soc.,* (70) 1, 1964, pp. 165-172.