The Borel-Weil Theorem and the K-Theory of Compact Lie Groups

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Abstract

The K-theory of a compact Lie group with torsion-free fundamental group is an exterior algebra on the vector bundles associated with the fundamental irreducible representations. The Borel-Weil theorem gives a characterization of these irreducible representations in terms of sections of holomorphic line bundles, which allows their calculation. The first section of the paper contains general representation theory of compact Lie groups. The second part contains the proof of the Borel-Weil theorem whereas the last part contains the actual construction of $K^*(G)$ for G a compact Lie group with torsion-free fundamental group. In the end we present an example.

1 Introduction

Let G be a connected compact Lie group and T its maximal torus. We write $\mathfrak{g}$ (resp. $\mathfrak{g}_{\mathbb{R}}$) and $\mathfrak{t}$ (resp. $\mathfrak{t}_{\mathbb{R}}$) for their complexified (resp. real) Lie algebras. Let L be the unit lattice, i.e., the kernel of the exponential map from $\mathfrak{t}$ to T. The weight lattice is $\hat{T} \cong \Lambda = \{\lambda \in i\mathfrak{t}_{\mathbb{R}}^* | e^{\langle \alpha, L\rangle} \subset 2\pi i\mathbb{Z}\}$. In the context of the adjoint representation of G in $\mathfrak{g}$, let Φ be the set of nonzero weights, i.e., the set of *roots*.

By complete reducibility of the adjoint representation we may write

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}^{\alpha} \right) = \mathfrak{n}_{-} \oplus \mathfrak{t} \oplus \mathfrak{n}_{+}$$

where $\mathfrak{g}^{\alpha} = \{X \in \mathfrak{g} | [X, Y] = \langle \alpha, Y\rangle Y, \forall Y \in \mathfrak{t}_{\mathbb{R}}\}$. Here $\mathfrak{n}_{+} = \bigoplus_{\alpha \in \Phi^{+}} \mathfrak{g}^{\alpha}$ is a Lie subalgebra ($\mathfrak{n}_{-} = \bar{\mathfrak{n}}_{+}$).

An element $X \in i\mathfrak{t}_{\mathbb{R}}$ is called regular if it is not a zero of any root. Let Φ^{+} be the set of positive roots associated with a regular element. A root is called simple if it is in Φ^{+} and if it cannot be written as sum of positive roots. The set of simple roots is denoted by Ψ.

For any α there is a choice of $E_{\alpha} \in \mathfrak{g}^{\alpha}, E_{-\alpha} \in \mathfrak{g}^{-\alpha}, H_{\alpha} \in i\mathfrak{t}_{\mathbb{R}}^*$ so that $[E_{\alpha}, E_{-\alpha}] = H_{\alpha}, [H_{\alpha}, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}$.

If $W = N_G(T)/T$ is the Weyl group then there exist $s_{\alpha} \in W$ so that $s_{\alpha} H_{\alpha} = -H_{\alpha}, s_{\alpha} H = H, \forall (H, H_{\alpha}) = 0$.

The symmetries given by the simple roots generate the Weyl group. Also, any root β can be written (by construction of Ψ) as

$$\beta = \alpha_1 + \cdots + \alpha_s \tag{1}$$

so that the partial sums $\alpha_1 + \cdots + \alpha_i$ are roots.

The Weyl groups acts simply tranzitively on the set of Weyl chambers (connected components of the regular elements). The maximal Weyl chamber is therefore a (nonessential) choice

$$\mathcal{C}^* = \{\lambda \in i\mathfrak{t}_{\mathbb{R}}^* | \langle \lambda, \alpha\rangle > 0\}$$

I state three results that are going to be used in the subsequent sections of the paper.

Proposition 1 *(Chevalley)The set* $\{w \in W | w\lambda = \lambda\}$ *is generated by the symmetries* $\{s_{\alpha} | \langle \lambda, \alpha\rangle = 0\}$.

For a proof see [Kna86].

The proof of the Borel-Weil theorem relies heavily on the following two theorems. Proofs may be found in [Kna88, Var84, BtD95].

Theorem 2 *(Theorem of the Heighest Weight) Let* (π, V) *be an irreducible representation of* G, λ *a weight of* π *and let* $v_0 \in V^{\lambda}$ *(i.e., the eigenspace of* λ*) a nonzero element. Then the following are equivalent:*

1. $\lambda + \alpha$ *is not a weight for any positive root* α.

2. *$\pi(\mathfrak{g}^\alpha)v_0 = 0$ for any positive root α.*

3. *Every weight of π can be expressed as λ minus a nonnegative linear combination of positive roots.*

In this case λ is called a heighest weight and $\lambda \in \Lambda \cap \overline{\mathcal{C}}^*$, $\dim V^\lambda = 1$. Via the Weyl character formula, λ determines π uniquely up to isomorphism.

Theorem 3 *(Peter-Weyl) Let G be a compact Lie group and let $\hat{G}$ be the set of isomorphism classes of irreducible unitary representations of G. (The theorem of heighest weight shows that $\hat{G} \cong \Lambda \cap \overline{\mathcal{C}}^*$.) Then*

$$L^2(G) \cong \widehat{\bigoplus_{i \in \hat{G}}} V_i \otimes V_i^*,$$

(in the sense of Hilbert space completion).

2 The Borel-Weil Theorem

2.1 Holomorphic Line Bundles

Fix a root λ. Define $\mathbb{C}_\lambda$ to be the representation space of the one-dimensional representation of T on $\mathbb{C}$ via e^λ.

Definition 1 *The holomorphic line bundle associated to λ is the fiber product $\mathcal{L}_\lambda = G \times_T \mathbb{C}_\lambda$ $(= G \times \mathbb{C}_\lambda / \{(g, z) \sim (gt^{-1}, e^\lambda(t)z) | t \in T\})$.*

Note that the $T \times \mathbb{C}_\lambda$ bundle $G \times \mathbb{C}_\lambda \longrightarrow G/T$ induces a $\mathbb{C}_\lambda$ bundle $\mathcal{L}_\lambda \longrightarrow G/T$. Therefore $\mathcal{L}_\lambda$ has the structure of a differentiable complex line bundle. Left translation on G/T lifts to left translation on $\mathcal{L}_\lambda$.

The map $\pi : G \longrightarrow G/T$ has local differentiable section so we get the following:

Remark 1 1. *The maximal torus T acts on the fiber at the identity via e^λ.*

2. *The differential structure on an open set $U \subset G/T$ is given by*

$$\mathcal{C}^\infty(U, \mathcal{L}_\lambda) = \{f \in \mathcal{C}^\infty(\pi^{-1}U) | f(gt) = e^\lambda(t) f(g), \forall t \in T\},$$

The first remark is obvious while the second one follows from the first one and the action of G on $\mathcal{L}_\lambda$.

We have a good description of differentiable functions with values in the holomorphic line bundle, but we would like a description of the holomorphic functions instead. These are the ones that satisfy the Cauchy-Riemann equation, and on the level of infinitesimal operators, these amount to annihilation by right action by $\mathfrak{n}_-$.

The space of holomorphic maps on U is

$$\mathcal{O}(U) = \{f \in \mathcal{C}^\infty(\pi^{-1}U) | f(gt) = f(g), \forall t \in T, r(X)f = 0, \forall X \in \mathfrak{n}_-\}.$$

The space of holomorphic maps on U with values in the holomorphic line bundle is given by the following proposition (which also proves that $\mathcal{L}_\lambda$ has a holomorphic structure.

Proposition 4 *The bundle $\mathcal{L}_\lambda$ has a complex structure which is compatible with its differentiable structure so that G acts via holomorphic bundle maps. Holomorphic maps on U with values in the bundle are*

$$\{f \in \mathcal{C}^\infty(\pi^{-1}U) | f(gt) = e^\lambda(t) f(g), \forall t \in T, r(X)f = 0, \forall X \in \mathfrak{n}_-\}.$$

Proof: To prove this it is more effective to use notation imported from cohomology (or more specifically in my case, from [GS69]). Note that $\mathcal{C}^\infty(U, \mathcal{L}_\lambda) = (\mathcal{C}^\infty(\pi^{-1}U) \otimes \mathbb{C}_\lambda)^T$, where the exponent means that we take invariants under the action of the exponent. Moreover we have that $\mathcal{O}(U, \mathcal{L}_\lambda) = (\mathcal{C}^\infty(U, \mathcal{L}_\lambda))^{\mathfrak{n}_-}$. Local isomorphism with $\mathcal{O}(U)$ induces the complex holomorphic structure that we want. ■

2.2 Holomorphic maps and Irreducible Representations

This section contains the statement of the Borel-Weil theorem. It is worth noting that this follows directly from the theorem of the heighest weight.

Theorem 5 *(Borel-Weil) Let $\lambda \in \Lambda$. Then $\mathcal{O}(G/T, \mathcal{L}_\lambda)$ is*

1. *0 is $\lambda \notin \Lambda \cap \overline{\mathcal{C}}^*$.*

2. *For $\lambda \in \Lambda \cap \overline{\mathcal{C}}^*$ it is non-zero finite dimensional and the resulting representation of G is irreducible of heighest weight λ.*

Proof: We have the following characterization of holomorphic maps:

$$\mathcal{O}(G/T, \mathcal{L}_\lambda) = \{f \in \mathcal{C}^\infty(G) | r(t)f = e^{-\lambda}(t) f, \forall t \in T, r(X)f = 0, \forall X \in \mathfrak{n}_-\}.$$

(The minus sign in the exponent comes from the right action.)

Since Fourier series may be differentiated term by term we may use the Peter-Weyl theorem to get a characterization of holomorphic maps:

$$\mathcal{O}(G/T, \mathcal{L}_\lambda) = \widehat{\bigoplus_{i \in \hat{G}}} V_i \otimes \{v^* \in V_i^* | \pi_i^*(t) v^* = e^{-\lambda}(t) v^*, \forall t \in T, \pi_i^*(X) v^* = 0, \forall X \in \mathfrak{n}_-\}.$$

Choose an i so that the tensor product is not zero. Then $v^* \neq 0$. The first condition on v^* says that $v^* \in (V_i^*)^{-\lambda}$. This is the place where we apply the theorem of the heighest weight. Since the eigenvalue is with a minus sign, we want to apply the theorem to $-\Psi^+$ instead of Ψ^+.

The second condition on v^* is that $\pi_i^*(X) v^* = 0, \forall X \in \mathfrak{n}_-$. Since $\mathfrak{n}_-$ runs over $-\Psi^+$ it means that the hypothesis of the heighest weight is satisfied.

1. If $\lambda \notin \Lambda \cap \overline{\mathcal{C}}^*$ then $(V_i^*)^{-\lambda} = 0$.

2. If $\lambda \in \Lambda \cap \overline{\mathcal{C}}^*$ then $\dim(V_i^*)^{-\lambda} = 1$. Then $-\lambda$ is the heighest weight of π_i^* so λ is the heighest weight of π_i.

■

Remark 2 *Note that* $\mathcal{O}(G/T, \mathcal{L}_\lambda) = H^0(G/T, \mathcal{O}(\mathcal{L}_\lambda))$*, in terms of sheaf cohomology, since they both represent holomorphic sections. There is a more general theorem, due to Bott, which characterizes higher cohomology groups in terms of actions of* W *on the higheste weights. See for example [AV00].*

3 K-Theory of G

Let G be a compact Lie group with $\pi_1(G)$ torsion-free, T the maximal torus. Consider $\rho_1, \rho_2, \ldots, \rho_l$ be the fundamental irreducible representations with maximal weights $\lambda_1, \lambda_2, \ldots, \lambda_l$ (meaning that λ_i are the simple weights in the dominant Weyl chamber).

This section follows the proofs in [Ati65].

3.1 K^*

We are dealing with the K functor in the sense of Grothendieck, i.e., formal differences of vector bundles over G. One may define $K^1(G)$ as $K(SG)$ where SG is the suspension, or alternatively, $[G, U]$, where U is the unitary group. Then one can define $K^*(G) = K(G) \oplus K^1(G)$ as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra due to Bott periodicity.

For ρ an irreducible representation of G, when interpreted as $\rho : G \longrightarrow U(n) \longrightarrow U$ for some n, one gets an element of $K^1(G)$. Alternatively, consider $G \circ G \longrightarrow SG$, G- principal bundle, where $G \circ G = \{ag_1 \oplus (1-a)g_2\}$ is the topological join (as defined in [Mil56]). The representation ρ induces a vector bundle V_ρ associated with this principal G-bundle. Then $k(\rho) = [V_\rho] - [\underline{\mathbb{R}}^{\dim \rho}] \in K^1(G)$ is the associated K-theory.

One has the notion of Chern character on K-theory, a ring isomorphism from $K^*(G) \otimes \mathbb{Q} \longrightarrow H^*(G, \mathbb{Q})$.

Lie groups are parallelizable, and they have a Spin structure, which implies that (via the Thom isomorphism theorem) we get a push-forward map

$$f_* : K^*(G) \longrightarrow K^*(\mathrm{pt}) = \mathbb{Z}$$

given by $f_*(x) = \langle ch(x)Td(G), [G] \rangle$, where Td is the Todd class. In the case of Lie groups, this is just one, so $f_*(x) = ch(x)[G]$.

3.2 Structure of K^* for $G, T, G/T$

3.2.1 Kunneth formulas

Let $p : G/T \times T \longrightarrow G$ be the conjugation map, $p(\bar{g}, t) = \bar{g}t\bar{g}^{-1}$. Using the Lefschetz fixed point theorem for left translation on G/T together with a calculation of $\det(1 - Adg)$ yeilds that p has degree $\chi(G/T) = |W|$.

This allows us to relate $K^*(G)$ to $K^*(G/T \times T)$. Atiyah ([Ati62]) proves that for K-theory there is a Kunneth formula

$$0 \longrightarrow K(X) \otimes K(Y) \longrightarrow K(X \times Y) \longrightarrow Tor(K(X), K(Y)) \longrightarrow 0.$$

Remark 3 *Since all irreducible representations of T are one dimensional, $K^*(T)$ is an exterior algebra on $k'(\lambda_i)$, where the $k'(\lambda_i) \in K(G/T)$ are associated to λ_i as vector bundles.*

Since this algebra is torsion-free, the Kunneth formula above is equivalent to $K^*(G/T \times T) = K^*(G/T) \otimes K^*(T)$.

3.2.2 G/T

Therefore, in order to get information about $K^*(G)$ we need to understand, on the one hand the structure of $K^*(G/T)$ and on the other hand how it interacts with $K^*(T)$. We will not get full descriptions, but we will use the Chern character to get sufficient information to find $K^*(G)$.

The following proposition gives the relation between the three K-theories.

Proposition 6 *Let ρ be a representation of G with weights $\mu_1, \mu_2, \ldots, \mu_n$. Then*

$$p^* k(\rho) = \sum k'(\mu_j) \otimes k(\mu_j).$$

Proof: Consider the following diagram of principal bundles:

$$\begin{tikzcd}
G \times (T \circ T) \arrow[r] \arrow[d, "\theta"] & G \circ G \arrow[d, "\psi"] \\
G/T \times ST \arrow[r, "f"] \arrow[d, "h"] & (G \circ G)/T \arrow[d, "g"] \\
S(G/T \times T) \arrow[r, "Sp"] & SG
\end{tikzcd}$$

The bottom map is the suspension of p, the top map is induced by left translation. All the other maps are constructed in the obvious way, which makes the diagram commute (ψ is projection, θ is projection together with the natural projection on joins, etc).

Consider t the principal T-bundle $G \circ G$. Then $f^*\psi(t)$ is a principal T-bundle on $G/T \times ST$. The $T \times T$ bundle $G \times (T \circ T) \longrightarrow G/T \times ST$ is associated to $f^*\psi(t)$ via translations (left the same as right since T is abelian). So for $\lambda \in \hat{T}$ take $\psi(\lambda) \in K((G \circ G)/T)$ associated to λ (for bundles over T we called this k').

Then $f^*\psi(\lambda)$ is associated to the $T \times T$-bundle which is associated to the representation. Therefore, it will be $k'(\lambda)$ for the bundle $G \longrightarrow G/T$ and V_λ for the bundle $T \circ T \longrightarrow ST$ by constructions. Therefore $f^*\psi(t) = k'(\lambda) \otimes [V_\lambda] \in K(G/T) \otimes K(ST)$.

Now, all representations of T are 1-dimensional, which means that V_λ has rank 1. Therefore $k(\lambda) = [V_\lambda] - [\mathbb{R}]$ so

$$f^*\psi(t) = k'(\lambda) \otimes (1 + k(\lambda)).$$

From here onwards all comes down to (general) nonsense:

$$h^*(Sp)^*[V_\rho] = f^*g^*[V_\rho] = \sum f^*\psi(\mu_i),$$

(this last equality is formal, since ρ has weights μ_i. Now this is equal by the above to:

$$\sum k'(\mu_i) \otimes (1 + k(\mu_i)).$$

By construction of associated vector bundles $\sum k'(\mu_i)$ will be the vector bundle over G/T whose fiber is the space of the representation ρ everywhere (since μ_i are all the weights of the representation ρ). Therefore $\sum k'(\mu_i) = [\mathbb{R}^{\dim \rho}]$.

This implies the formula in the statement of the lemma by noting that h^* is injective (since it collapses endpoints).

■

3.2.3 *T*

The next proposition we need is concerned with the more intimate relation between the representations and $K^*(T)$. Let $R(T)$ be the representation ring, i.e., the integral ring of $\hat{T}$.

Let $\Delta = \sum_{w \in W} \varepsilon(w) e^{w \sum \lambda_i} \in R(T)$.

Proposition 7 *If ρ_i has weights λ_{ij} then*

$$\prod_1^l \left(\sum \lambda_{ij} \otimes k(\lambda_{ij})\right) = \Delta \otimes \prod k(\lambda_i) \in R(T) \otimes K^*(T).$$

Proof: We saw that $K^*(T)$ is an exterior algebra on l generators so $\prod_1^l \left(\sum \lambda_{ij} \otimes k(\lambda_{ij})\right) = \prod_1^l \left(\sum \lambda_{ij} \otimes \sum N_{ijk} k(\lambda_k)\right) = \sum \alpha \otimes \prod k(\lambda_i) = A \otimes \prod k(\lambda_i)$ since whenever the second product has repeating terms, it vanishes.

There is an induced action of the Weyl group W on $R(T)$. W also acts on $K^*(T)$ by acting on the generators: $wk(\lambda_i) = k(w\lambda_i)$. This induces an action on $R(T) \otimes K^*(T)$ which permutes the weights of the representations ρ_i. The quantity on the LHS of the relation is permutation invariant, so $A \otimes \prod l(\lambda_i)$ is W-invariant.

Therefore $w \in W \implies A \otimes \prod k(\lambda_i) = wA \otimes \prod k(\lambda_i) = (wA) \otimes w \prod k(\lambda_i) = (wA) \otimes \prod k(w\lambda_i) = \varepsilon(w)(wA) \otimes \prod k(\lambda_i)$, since $\varepsilon(w)$ is actually the sign of the permutation, and we are in an exterior algebra. Therefore $wA = \varepsilon(w)A$.

Therefore A is W-alternating. If W^+ is the dominant Weyl chamber then $\{\Delta_\lambda | \lambda \in W^+\}$ form a basis for the W-alternating characters, where $\Delta_\lambda = \sum_{w \in W} \varepsilon(w) e^{w\lambda}$.

We would like to prove that $A = \Delta$. Atiyah's argument is somewhat incomprehensible, so I use [Kna86]. For now all we know is that A is some (integer) linear combination of Δ_λ for λ weights in the dominant Weyl chamber (since W acts simply tranzitively on the Weyl chambers).

Every such weight is a nonnegative integer linear combination of the maximal weights λ_i. If the coefficient of λ_k is 0, then Chevalley's theorem says that $\{w \in W | w\lambda = \lambda\}$ is generated by $\{s_\alpha | \langle \lambda, \alpha \rangle = 0\}$ which contains $w_0 = s_{\lambda_k}$. Therefore we have the obvious relation $\Delta_\lambda = \varepsilon(w_0)\Delta_{w_0\lambda} = -\Delta_\lambda$ since $\mathfrak{s}_\alpha$ are symmetries. Therefore $\Delta_\lambda = 0$.

In the expression of A, we have an integer linear combination of $e^{\sum \lambda_{\alpha\beta}}$. Since $\lambda_{\alpha\beta} \leq \lambda_\alpha$ with equality occuring for exactly one β it means that λ in the Δ_λ in the expansion of A are all of the form $\sum n_i\lambda_i \leq \sum \lambda_i$ with equality only for one λ. Therefore for exactly one case $\lambda = \sum \lambda_i$ the coefficient of at least one λ_i is nonzero so the $\Delta_\lambda \neq 0$. Therefore $A = \Delta_{\sum \lambda_i} = \Delta$ after a coefficient comparison.

■

3.2.4 G

Now we are ready to use the previous two lemmas to get a global result. Consider the product bundle $\alpha = \prod k(\rho_i)$. Then we have the following.

Proposition 8 *For α defined above we have $\langle ch\alpha, [G] \rangle = 1$, where ch is the Chern character.*

Proof: In order to prove this lemma, we need to use a formula that we will not prove, due to the fact that its proof is of a completely different nature than the focus of this paper. Since $\chi(G/T) = |W|$, formulas (20.1) and (20.3) in [BH59] give that $\langle \mathrm{ch}k'(\Delta), [G/T] \rangle = \chi(G/T) = |W|$.

The rest of the proof is reduced to computations:

$$p^*\alpha = p^* \prod k(\rho_i) = \prod p^*k(\rho_i).$$

But by proposition 6 $p^*k(\rho_i) = \sum_j k'(\lambda_{ij}) \otimes k(\lambda_{ij})$. Therefore

$$p^*\alpha = \prod_i (\sum_j k'(\lambda_{ij}) \otimes k(\lambda_{ij})) = (k' \otimes 1)(\Delta \otimes \prod k(\lambda_i),$$

by proposition 7.

Therefore

$$\langle \mathrm{ch}p^*\alpha, [G/T \times T] \rangle = \langle \mathrm{ch}k'(\Delta) \otimes \prod k(\lambda_i), [G/T \times T] \rangle = \langle \mathrm{ch}k'(\Delta), [G/T] \rangle \langle \mathrm{ch} \prod k(\lambda_i), [T] \rangle.$$

By the result at the beginning of this section, $\langle \mathrm{ch}k'(\Delta), [G/T] \rangle = |W|$. Moreover, $\langle \mathrm{ch} \prod k(\lambda_i), [T] \rangle = 1$ because $K^*(T)$ is the exterior algebra on $k(\lambda_i)$ and $\prod k(\lambda_i)$ is therefore a maximal element of the K-theory, which means that it is mapped to 1 via the Chern character (no other possibility, since ch is injective).

The proposition follows from here. ■

3.3 Splitting Principle

In order to determine $K^*(G)$ we need to make use of the "splitting principle filosophy". This means that we pull back vector bundles on G to vector bundles on $G/T \times T$ and then prove a certain injectivity property on the level of K-theory.

3.3.1 **Tors**$(K^*(G))$

One of the main problems of our approach to the calculation of $K^*(G)$ is the use of the Chern character to obtain injectivity relations on the level of K-theory, i.e., in the splitting principle. The Chern character is an isomorphism $K^* \otimes \mathbb{Q} \longrightarrow H^*$, but $\otimes\mathbb{Q}$ cancels torsion.

Therefore we assume that $\mathrm{Tors}K^*(G) = 0$ and refer the reader to [Hod67] for a proof of the fact that a torsion-free fundamental group implies a torsion-free K-theory.

3.3.2 $K^*(G)$

We have assumed that this ring is torsion-free. Therefore it is isomorphic to $H^*(G, \mathbb{Q})$. It is a standard result in Lie group theory (or alternatively, one can use the de Rham theorem to prove it, see for example [Bre93]) that $H^*(G, \mathbb{Q})$ is an exterior algebra on l generators, where l is the number of fundamental irreducible representations of G.

The main result is the following.

Theorem 9 *$K^*(G) = \Lambda k(\rho_i)$.*

Proof: Consider $j : \Lambda(1, \ldots, l) \longrightarrow K^*(G)$ defined by $j(i) = k(\rho_i)$. In order to prove the theorem we need to show that j is bijective.

Since $rk\Lambda = 2^l = rkH^*(G, \mathbb{Q}) = rkK^*(G)$. Therefore $[K^*(G) : Imj] < \infty$. The diagonal map of the diagram

$$\begin{tikzcd}
\Lambda \arrow[r, "j"] \arrow[dr, "i"'] & K^*(G) \arrow[d, "x \mapsto \{y \mapsto f_*(xj(y))\}"] \\
& \mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{Z})
\end{tikzcd}$$

is an isomorphism. This is true because for any $x \in \Lambda$, we may choose y to be so that $xy = \prod i \in \Lambda$. Then $i(x)(y) = f_*(j(x)j(y)) = f_*(\prod k(\rho_i)) = \langle \mathrm{ch} \prod k(\rho_i), [G] \rangle = 1$ by the previous proposition. Therefore x is never mapped to the trivial map.

This proves injectivity.

Now Imj has finite index in $K^*(G)$ which is torsion-free, and by bijectivity of i it means that Imj is a direct summand of $K^*(G)$ and therefore $Imj = K^*(G)$. This gives surjectivity. ■

4 Computations

We may use the Borel-Weil theorem to compute the irreducible representations of a compact Lie group G, which allows us to compute generators for $K^*(G)$, when G has torsion-free fundamental group.

For the purposes of this section, the statement of the Borel-Weil theorem in [Kna86] seems more appropriate. The irreducible representation of highest weight λ is given by $\{F : G \longrightarrow \mathbb{C}\}$ so that F is holomorphic and $F(xb) = e^{\lambda}(t)F(x), \forall t \in B$ where B is the Lie group corresponding to $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}_-$.

Since $\pi_1(SU(n)) = \pi_1(SU) = \pi_1(U) = \mathbb{Z}$ is torsion-free it means that we may apply these methods to compute $K^*(SU(n))$.

4.1 Irreducible Representations of $SU(n)$

4.1.1 The case $SU(2) = S^3$

The maximal torus is $U(1)$ and $G/T = SL(2,\mathbb{C})/B = \mathbb{CP}^1$.

In this case $B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \right\}$. The fundamental irreducible representation corresponds to the simple weight 1 in the dominant Weyl chamber. Then $\mathcal{O}(G/T, \mathcal{L}_\lambda)$ corresponds to holomorphic functions on $SL(2,\mathbb{C})$ on which B acts as multiplication by the top left element in the matrix form. These correspond to degree 1 homogeneous polynomials in two variables. On the level of $G = SU(2)$, these polynomials acts as multiplication by the column vector of coefficients.

Therefore $K^*(S^3)$ is an exterior algebra on one generator. From Bott periodicity, this generator should be the square of the Bott generator of $K(S^2)$. Therefore we have a characterization of $k(\rho)$ in terms of the standard Bott generator.

4.1.2 The case $SU(n)$

The maximal torus is $U(n-1)$ and then $G/T = SU(n)/U(n-1) = \mathbb{CP}^{n-1} = SL(n,\mathbb{C})/P$, where P is the set of matrices for which the first column has only 0-s except for the top left element. Holomorphic sections again correspond to homogeneous polynomials in n variables this time, and the representation is given by matrix multiplication, for the simple root 1.

However, these are not the only fundamental representations of $SU(n)$. This is the case for $n = 3$ but in general one needs to look at holomorphic sections of line bundles over Grassman manifolds.

5 Remarks

1. Independent of the results in [Hod67], this paper proves that $K^*(G)/\mathrm{Tors}K^*(G) = \Lambda k(\rho_i)$ just by replacing equality with equality $\pmod{\mathrm{Tors}K^*(G)}$ in the proof of the theorem.

2. Since S^1 is a compact Lie group, the structure theorem for $K^*(S^1)$ seems that this theory circumvents the standard proof for Bott's periodicity. This is not the case, since the notion of K^* already assumes the periodicity.

3. One could try to apply the same methods to get a KO-theory of compact Lie groups. The methods exhibited in this paper would fail, since the Kunneth formulas are not characteristic of general cohomology theories. For example, the Kunneth formulas for cohomology and K-theory have very different proofs, and according to Atiyah ([Ati62]), there is not Kunneth formula for KO-theory.

4. The results in this paper deal with compact Lie groups with torsion-free fundamental groups. In that case the torsion of the K-theory is 0. In general, this is not so, and the structure of $K^*(G)$ is much more complicated.

   In [HS73] the case of finite fundamental group for G is treated. For example, when $\pi_1(G) = \mathbb{Z}/p\mathbb{Z}$ for p prime one needs to pass to the universal Lie covering of G and tensor the exterior algebra with the quotient of the representation rings of the Lie cover and that of the fundamental group.

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