Hodge Theory of Kähler Manifolds

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Abstract

For differentiable complex manifolds the theory of elliptic operators and harmonic forms relate cohomology groups of complexes to the harmonic spaces associated with Laplacian operators. For compact Kähler manifolds, there is a relation between the Laplacian operators Δ and $\overline{\Box}$ which leads to a decomposition of the cohomology of differential forms as a direct sum of cohomologies of differential forms of (p, q) type (Hodge decomposition). This theory may be restricted to the case of Riemann surfaces and get important results, such as Kodaira-Serre duality, Riemann-Roch.

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1 Sheafitiation

1.1 Introduction

Let X be a topological space.

Definition 1 *A* **presheaf** $\mathcal{F}$ *of abelian groups (rings, modules) on X is a map from the topology on X to the category of abelian groups (rings, modules) with the following property:*

For opens $V \subset U$ there exist restriction functions $|_V = \rho_{U,V} : \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$ so that for any sets $W \subset V \subset U$ we have $\rho_{U,W} = \rho_{U,V} \circ \rho_{V,W}$. Moreover, $\rho_{U,U} = 1_U$.

Elements of $\mathcal{F}(U)$ are called sections of $\mathcal{F}$ over U.

Definition 2 *A presheaf $\mathcal{F}$ is called a* **sheaf** *is the following two conditions hold:*

1. *If f is a section over U and g is a section over V such that $f|_{U\cap V} = g|_{U\cap V}$ then there exists a section h over $U \cup V$ such that $f = h|_U$ and $g = h|_V$.*
2. *If f and g as before are the zero sections then h is also the zero section.*

Definition 3 *Let $\mathcal{F}$ be a sheaf on X. For $x \in X$ define the* **stalk** *at x to be*

$$\mathcal{F}_x = \varinjlim \mathcal{F}(U),$$

the limit taken over all neighborhoods of x.

Elements of the stalk are called germs.

Definition 4 *A map of sheaves $f : \mathcal{F} \longrightarrow \mathcal{G}$ is a morphism of presheaves, in a categorical sense. A sequence*

$$\mathcal{F} \xrightarrow{f} \mathcal{G} \xrightarrow{g} \mathcal{H},$$

is called **exact** *at $\mathcal{G}$ if:*

1. $gf = 0$.
2. *For any $p \in X$ and neighborhood U of p, if h is a section of $\mathcal{G}$ on U so that $gh = 0$ then there exists a neighborhood $V \subset U$ of p such that $h|V = fk|V$ for a section k of $\mathcal{F}$ on V.*

If X is compact then exactness is equivalent to exactness on stalks.

1.2 Examples

Sheaves are in some sense the next step up from vector bundles.

1. The sheaf of locally constant functions $\underline{\mathbb{C}}$.
2. The sheaf $\mathcal{S}$ defined as $\mathcal{S}(U) = \prod_{p \in U} G_p$, where G_p are abelian groups associated with p. If all the groups G_p are equal we get the sheaf of functions $X \longrightarrow G$. If all are trivial except for one, we get the skyscraper sheaf.
3. The divisor sheaf $Div(X)(U) = \bigoplus_{p \in U} \mathbb{Z}$.
4. Sheaf of holomorphic functions $\mathcal{O}_X$, sheaf of holomorphic p-differential forms Ω_X^p.

1.3 Cech Cohomology

As usually with cohomology, there is a functorial cohomology based on resolutions. However, from the point of view of computations, Cech cohomology is most useful.

Let $\mathcal{F}$ be a sheaf on the space X.

Let $\mathcal{U} = \{U_\alpha\}$ be an open cover of X. For a multiset I with k indexes write $U_I = \bigcap_{s \in I} U_s$. Define the k-th cochain complex to be $C^k(\mathcal{U}) = \prod_I \mathcal{F}(U_I)$. For $f \in C^k(\mathcal{U})$ and a k-multiset I let f_I be the projection to $\mathcal{F}(U_I)$.

Define the coboundary operators $d_k : C^k(\mathcal{U}) \longrightarrow C^{k+1}(\mathcal{U})$ as $d_k f = g$, such that if I is a $(k+1)$-set then

$$g_I = \sum_{i=0}^{k} (-1)^i f_{I \setminus \{i\}}|_{U_I},$$

Let $H^*(\mathcal{U}, \mathcal{F})$ be the cohomology of this complex. The natural ordering of coverings means that we can make sense of a cohomology of X as

$$H^*(X, \mathcal{F}) = \varinjlim H^*(\mathcal{U}, \mathcal{F}).$$

The following properties are easy to prove

Proposition 5 *1. $H^0(X, \mathcal{F}) = \mathcal{F}(X)$. (Since $H^0(\mathcal{U}, \mathcal{F}) = \mathcal{F}(X)$ for all coverings $\mathcal{U}$.)*

2. $H^1(X, \mathcal{F}) = 0$ if and only if $H^1(\mathcal{U}, \mathcal{F}) = 0$ for all coverings $\mathcal{U}$.

General cohomology theory for $H^*(X, \mathcal{F})$ implies the following theorem ([Mir95]).

Proposition 6 *Let X be a paracompact space with an exact sequence of sheaves on it*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{H} \longrightarrow 0.$$

Then there exists a long exact sequence on the level of cohomology

$$\begin{array}{cl} 0 & \longrightarrow H^0(X,\mathcal{F}) \longrightarrow H^0(X,\mathcal{G}) \longrightarrow H^0(X,\mathcal{H}) \longrightarrow \\ & \longrightarrow H^1(X,\mathcal{F}) \longrightarrow H^1(X,\mathcal{G}) \longrightarrow H^1(X,\mathcal{H}) \longrightarrow \\ & \longrightarrow H^2(X,\mathcal{F}) \longrightarrow H^2(X,\mathcal{G}) \longrightarrow H^2(X,\mathcal{H}) \longrightarrow \ldots \end{array}$$

Definition 7 *The sheaf $\mathcal{F}$ on X is called* **fine** *if there exist partitions of unity. More precisely, if for any covering $\mathcal{U}$ of U there are maps $f_\alpha : \mathcal{F}(U_\alpha) \longrightarrow \mathcal{F}(U)$ such that the support of $f_\alpha(x)$ is contained in U_α and $\sum f_\alpha(x|_{U_\alpha}) = x$.*

Lemma 8 *Let $\mathcal{F}$ be a fine sheaf and X paracompact. Then for all $k > 0$ we have $H^k(X,\mathcal{F}) = 0$.*

Proof: Let $\mathcal{U}$ be a locally finite cover. Let x be a closed k cochain on the Cech cochain complex defined for the cover $\mathcal{U}$.

Define y a $(k-1)$ cochain as follows:

For each I a $(k-1)$-multiset of indexes set

$$y_I = \sum_{i \in I} f_i x_{i \cup I},$$

where f_i is a partition of unity and the terms in the sum extend by 0 to all of X.

It is easy to see that $dy = x$. ■

2 Cohomology on Complex Manifolds

Let X be a complex manifold, T_z^*X be the cotangent space at $z \in X$. The complex structure on T_zX implies the existence of a decomposition

$$T_z^*X = T_z^{*1,0}X \oplus T_z^{*0,1}X,$$

into the plus and minus eigenspaces of the complex structure.

In terms of local coordinates $z_1, \ldots, z_n$ at p, the cotangent space $T_z^{*1,0}X$ is spanned by $dz_1, \ldots, dz_n$ (it is called the space of holomorphic 1-forms) while the space $T_z^{*0,1}X$ is spanned by $d\bar{z}_1, \ldots, d\bar{z}_n$ (it is called the space of antiholomorphic 1-forms).

Definition 9 *Let $\mathcal{E}^{p,q}$ be the sheaf $\wedge^p T^{*1,0}X \otimes \wedge^q T^{*0,1}X$. Let $\mathcal{E}^r$ be the sheaf $\wedge^r T^*X$.*

The operators $\partial, \bar{\partial}$ on T_zX extend to sheaf operators $\partial, \bar{\partial}$ on $\mathcal{E}^{p,q}$.

Proposition 10 (Poincare $\bar{\partial}$ lemma) *On a complex manifold X we have the following exact sequence:*

$$0 \longrightarrow \Omega^p \longrightarrow \mathcal{E}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{E}^{p,1} \xrightarrow{\bar{\partial}} \ldots.$$

Definition 11 *Define $H^{p,q}(X) = H(X, \mathcal{E}^{p,q}(X))$ the Dolbeault cohomology groups of global (p,q)-forms.*

Note that even though the Dolbeault complex in the Poincare lemma is exact, exactness is on the level of sheaves. For global sections the sequence is no longer exact.

Theorem 12 (Dolbeault) *For X a complex manifold we have*

$$H^q(X, \Omega^p) = H^{p,q}(X).$$

Proof: Since $\mathcal{E}^{p,q}$ is fine Lemma 8 implies that $H^k(X, \mathcal{E}^{p,q}) = 0$.

Moreover, the Poincare $\bar{\partial}$-lemma gives the following short exact sequences (where $\mathcal{Z}^{p,q}$ stands for the sheaf of closed (p,q)-cochains:

$$\begin{array}{c} 0 \longrightarrow \Omega^p \longrightarrow \mathcal{E}^{p,0} \longrightarrow \mathcal{Z}^{p,1} \longrightarrow 0 \\ \vdots \\ 0 \longrightarrow \mathcal{Z}^{p,q} \longrightarrow \mathcal{E}^{p,q} \longrightarrow \mathcal{Z}^{p,q+1} \longrightarrow 0 \end{array}$$

These gives long exact sequences on cohomology (Proposition 6)

$$\begin{array}{lcl} \dots H^{q-1}(X, \mathcal{E}^{p,0}) & \longrightarrow & H^{q-1}(X, \mathcal{Z}^{p,1}) \longrightarrow H^q(X, \Omega^p) \longrightarrow H^q(X, \mathcal{E}^{p,0}) \dots \\ \dots H^{q-2}(X, \mathcal{E}^{p,1}) & \longrightarrow & H^{q-2}(X, \mathcal{Z}^{p,2}) \longrightarrow H^{q-1}(X, \mathcal{Z}^{p,1}) \longrightarrow H^{q-1}(X, \mathcal{E}^{p,1}) \dots \\ & \vdots & \\ \dots H^1(X, \mathcal{E}^{p,q-2}) & \longrightarrow & H^1(X, \mathcal{Z}^{p,q-1}) \longrightarrow H^2(X, \mathcal{Z}^{p,q-2}) \longrightarrow H^2(X, \mathcal{E}^{p,q-2}) \dots \\ H^0(X, \mathcal{E}^{p,q-1}) & \xrightarrow{\bar{\partial}} & H^0(X, \mathcal{Z}^{p,q}) \longrightarrow H^1(X, \mathcal{Z}^{p,q-1}) \longrightarrow H^1(X, \mathcal{E}^{p,q-1}) \dots \end{array}$$

Since the outter groups are trivial we get the isomorphisms

$$\begin{aligned} H^q(X, \Omega^p) &= H^{q-1}(X, \mathcal{Z}^{p,1}) = \dots = H^1(X, \mathcal{Z}^{p,q-1}) = H^0(X, \mathcal{Z}^{p,q}) / \bar{\partial} H^0(X, \mathcal{E}^{p,q-1}) \\ &= \mathcal{Z}^{p,q}(X) / \bar{\partial} \mathcal{E}^{p,q-1}(X) = H^{p,q}(X). \end{aligned}$$

■

3 Elliptic Operators

3.1 Differential Operators

Let X be a compact differentiable manifold.

Let $\sum g_{ij} dx_i \otimes dx_j$ be a Riemann metric on X. Consider the volume element $d\mu = \sqrt{|\det g_{ij}|} dx_1 \dots dx_n$. Let E be a Hermitian vector bundle on X.

Define $W^0(X, E)$ to be the completion of differentiable sections of E with respect to the L^2 norm $(d\mu)$. Using partitions of unity, we may define the s-Sobolev norm on sections of E in terms of the Sobolev norm on differentiable functions:

$$||f||_s^2 = \int |\hat{f}|^2 (1 + |y|)^s dy,$$

where $\hat{f}$ is the Fourier transform.

Let $W^s(X, E)$ be the completion of the space of sections with respect to this norm.

There are two important theorems regarding these spaces, proven in [Wel80] (4.1).

Proposition 13 (Sobolev) *If f is $L^2(\mathbb{R}^n)$ measurable with finite s-Sobolev norm such that $s > \lfloor n/2 \rfloor + k + 1$ then f is k times differentiable.*

Proposition 14 (Rellich) *If $t < s$ the inclusion operator $W^t \subset W^s$ is compact.*

Let X be a compact differentiable manifold and E, F complex vector bundles on X.

Definition 15 *An operator $L : E(X) \longrightarrow F(X)$ (global sections) is called a k-differential operator if when passing to local trivialisations we get a linear partial differential operator of order k. The space of k-differential operators is $Diff_k(E, F)$.*

Definition 16 *Let $OP_k(E,F)$ be the space of k-operators, i.e., the space of operators $L : E(X) \longrightarrow F(X)$ such that for all s there is a continuous extension $L_s : W^s \longrightarrow W^{s-k}$. Note that $Diff_k \subset OP_k$.*

Sobolev's lemma (13) together with existence of adjoints gives the following proposition:

Proposition 17 *For $L \in OP_k(E,F)$ there exists an adjoint operator $L^* \in OP_k(F,E)$ such that the extensions to W^s and respectively W^{s-k} are adjoint with respect to the Sobolev norm.*

Definition 18 *Let E^*, F^* be the pullbacks of E and F to the bundle of nonzero cotangent vectors. A k-* **symbol** *$\sigma : E^* \longrightarrow F^*$ is a linear map that satisfies $\sigma(u, xv) = x^k \sigma(u,v)$ for any $u \in X$ and $v \in E_u$. Let $Symb_k(E,F)$ be the space of symbols.*

Proposition 19 *There is a natural map $\sigma_k : Diff_k(E,F) \longrightarrow Symb_k(E,F)$.*

Proof: Let L be a k-differential operator. We define $\sigma_k(L)(u,v)$ at each $w \in E^*$. There exists a global differentiable function f such that $df|u = v$ and a global differentiable section g of E such that $g(u) = w$.

Define $\sigma_k(L)(u,v)w = L\left(i^k/k!(f - f(u))^k g\right)(u)$. It is not hard to see that this is independent of choices. ■

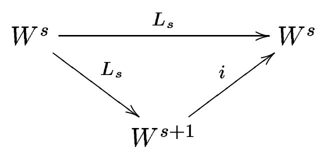
Note that since we evaluate the symbol at u, if the differential operator L has no k order terms, the symbol will be 0. The other directions is also true and the kernel of the σ_k map is $Diff_{k-1}(E,F)$.

An easy consequence of Proposition 19 is that $\sigma_1(\bar{\partial}) = i\pi_{0,1} v \wedge w$ for the Dolbeault complex. For the de Rham complex we have $\sigma_1(d) = iv \wedge w$.

We end this section with a proposition that will be useful to prove that certain operators are Fredholm.

Proposition 20 *If X is compact and $L \in OP_{-1}(E,E)$ then L is compact.*

Proof: By definition of OP_{-1} we have a commutative diagram



where i is the compact inclusion by Rellich's lemma (14).

Therefore L_s is a compact operator when of order 0. ■

3.2 Elliptic Operators

Let $L \in Diff_k(E,F)$.

L is called **elliptic** if $\sigma_k(L)(u,v)$ is an isomorphism on fibers.

I now use a theorem from the theory of pseudodifferential operators ([Wel80], 4.3). There exists a pseudodifferential operator L^{-1} such that $L^{-1}L - 1 \in OP_{-1}(E)$. Therefore this is compact. Then the operator $L^{-1}L$ is Fredholm (see for example [Ati89]) and so $\mathcal{K}_L = \ker L$ is finite dimensional. Moreover, by successive applications of the Sobolev lemma (13) (for a solution $x \in \mathcal{K}_L \iff x = (L^{-1}L - K)x$ where K is compact) we get that $\mathcal{K}_L \subset E(X)$.

Let $L \in Diff_k(E)$ be a self-adjoint elliptic operator.

Proposition 21 *There exist maps H_L, G_L acting on global sections of E such that*

1. *$ImH_L = \mathcal{K}_L$ and $LG_L + H_L = G_L L + H_L = 1$.*
2. *$H_L, G_L \in OP_0$.*
3. *$E(X) = \mathcal{K}_L \oplus G_L L(E(X))$.*

Proof: The map H_L is easy to define. Simply take projection to $\mathcal{K}_L$.

There is a continuous bijection ([Wel80], 4.4.4.11) from $W^k \cap \mathcal{K}_L^{\perp} \longrightarrow W^0 \cap \mathcal{K}_L^{\perp}$. The open mapping theorem gives an inverse G_0. Take $G_L = G_0|E(X)$.

Points 1 and 2 are now clear. Point 3 comes from the decomposition for self-adjoint operators. ■

3.3 Harmonic Theory and Cohomology

An elliptic complex is a sequence $\mathcal{E} = (E_k)$ of differentiable vector bundles with operators $L_k : E_k(X) \longrightarrow E_{k+1}(X)$ with the obvious condition $L^2 = 0$ and such that the associated sequence on the level of symbols is exact. Let $H^*(\mathcal{E})$ be the cohomology of this complex.

Define the **Laplacians** $\Delta_i = L_i^* L_i + L_{i-1} L_{i-1}^*$. These are maps of complexes.

Proposition 22 *The operators Δ_i are self-adjoint and elliptic.*

Proof: Self-adjointness is clear.
By definition of symbol, we have

$$\sigma_{2k}(\Delta_i) = \sigma_{2k}(L_i)^* \sigma_{2k}(L_i) + \sigma_{2k}(L_{i-1})\sigma_{2k}(L_{i-1})^* = s_i^* s_i + s_{i-1} s_{i-1}^*.$$

The ellipticity of the complex implies that we have a commutative diagram with both rows exact:

$$\begin{array}{ccccc} L_{i-1} & \xrightarrow{s_{i-1}} & L_i & \xrightarrow{s_i} & L_{i+1} \\ \downarrow & & \downarrow & & \downarrow \\ L_{i-1} & \xleftarrow[s_{i-1}^*]{} & L_i & \xleftarrow[s_i^*]{} & L_{i+1} \end{array}$$

It is clear from the diagram that $L_i = \mathrm{Im} s_{i-1} \oplus \mathrm{Im} s_i^*$. Now it is clear that $s_i^* s_i + s_{i-1} s_{i-1}^*$ is bijective on $\mathrm{Im} s_{i-1} \oplus \mathrm{Im} s_i^*$ for nonsensical reasons. ■

Therefore we may apply the results from the previous section. Set $\mathcal{K}(E_k) = \mathcal{K}_{\Delta_k}$. For simplicity write $E = \bigoplus E_k$. It is efficient to think of this as a grading.

Define $L = \oplus L_k$, $H = \oplus H_{\Delta_k}$, $G = \oplus G_{\Delta_k}$. Then we have

$$\begin{aligned} \Delta &= LL^* + L^*L \\ 1 &= H + G\Delta = H + \Delta G \\ HG &= GH = H\Delta = \Delta H = 0 \\ L\Delta &= \Delta L \end{aligned}$$

We continue with a technical lemma.

Lemma 23 *Let s be a global section of E. Then $\Delta s = 0$ if and only if $Ls = L^*s = 0$. This implies that $LH = HL = 0$ and $LG = GL$.*

Proof: With respect to the 0-Sobolev norm we have

$$\langle \Delta s, s \rangle = ||Ls||^2 + ||L^*s||^2.$$

Adjointness of H implies the second relation.

Since L and G are both 0 on $\mathcal{K}(E)$, it is enough to show the commutation relation on $\mathcal{K}(E)^\perp$. The first part of Theorem 24 implies that these elements are of the form Δs for $s \in GE(X)$.

This will follow from a series of equalities that use the relations metioned until now.

$$\begin{aligned} DL\Delta s - LG\Delta s &= G\Delta L s - LG\Delta s \\ &= (1-H)Ls - L(1-H)s = (LH - HL)s = 0 \end{aligned}$$

■

The following theorem makes the connection between harmonic theory and cohomology.

Theorem 24 *1. We have the orthogonal decomposition*

$$E(X) = \mathcal{K}(E) \oplus LL^*GE(X) \oplus L^*LFE(X).$$

2. We also have

$$\mathcal{K}(E_k) = H^k(\mathcal{E}).$$

Proof: The decomposition theorem for elliptic operators gives

$$E(X) = \mathcal{K}(E) \oplus (LL^*GE(X) + L^*LFE(X)).$$

But the adjointness property of L (recall that $L^2 = 0$) implies that this is a direct sum.

To prove the cohomological statement define $H : Z^k(\mathcal{E}) \longrightarrow \mathcal{K}(E_k)$ by just applying the map H to the corresponding node in the complex.

Surjectivity of H follows from $I = H + G\Delta$. We only need that $\ker H = B^k(\mathcal{E})$.

Assume that $H\sigma = 0$ and $L\sigma = 0$. We have the decomposition

$$\sigma = H\sigma + LL^*G\sigma + L^*LG\sigma.$$

Since $LG = GL$ from the previous lemma, we get that $\sigma = LL^*G\sigma$. Therefore σ is a coboundary. ∎

We will later apply this interpretation of cohomology to prove the Hodge decomposition theorem for Kähler manifolds where there are certain relations among these harmonic operators.

4 Compact Kähler Manifolds

4.1 Operators on Vector Spaces

Let X be a compact complex manifold. Let E be a hermitian vector space of complex dimension n.

Definition 25 *The* **Hodge star** *operator* $* : \wedge^k E \longrightarrow \wedge^{2n-k} E$ *such that* $x \wedge *x$ *is the volume form for any* x *basis element of* $\wedge^k E$.

Note that this implies that $x \wedge *y = \langle x, y\rangle v$, where v is the volume element, i.e., $v = e_1 \wedge \ldots \wedge e_{2n}$, for basis e_i.

Let $\wedge E = \bigoplus \wedge^k E$. Let p_k be projection onto the degree k space of this graded ring. Let $w = \sum(-1)^k p_k = **$. The choice of letter w is not random, since this represents the reflection of the Weyl group of $SU(2)$.

Let E^* be the complexification of the real dual space of E ($E^* = E^*_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$). There is an obvious decomposition $E^* = E^{*1,0} \oplus E^{*0,1}$. If we write $E^{p,q} = \wedge^p E^{*1,0} \wedge \wedge^q E^{*0,1}$ then we have the decomposition

$$\wedge E^* = \bigoplus_n \oplus_{p+q=n} E^{p,q}. \tag{1}$$

Consider the basis $z_1, \ldots, z_n$ of $E^{*1,0}$. Then $\bar{z}_1, \ldots, \bar{z}_n$ is a basis of $E^{*0,1}$. The hermitian inner product h on E is given in local coordinates by $h = \sum h_{ij} z_i \otimes \bar{z}_j$.

Definition 26 *The fundamental* $(1,1)$*-form associated with* h *is* $\Omega = -Imh/2 = \frac{i}{2}\sum h_{ij} z_i \wedge \bar{z}_j$.

Note that the operators $*$ and p_k extend to $\wedge E^*$. Let $p_{p,q}$ be projection to $E^{p,q}$ (formula 1). Define $J = \sum i^{p-q} p_{p,q} : \wedge E^* \longrightarrow \wedge E^*$, the complex structure operator on the complexified tangent space. Note that $J^2 = w$.

4.2 Operators on Complex Manifolds

Let X be a compact complex manifold. The operator $*$ extends to an operator on the exterior product of the cotangent bundle $\wedge T^*X$. This induces an isomorphism between differentiable sections of $\wedge^k T^*X$ and differentiable sections of $\wedge^{2n-k} T^*X$.

Note that $*1 = v$ is the volume element of X (where 1 represents the constant section of the trivial $\underline{\mathbb{C}}$ bundle on X). Define the Hodge star metric to be $\langle f, g\rangle = \int_X f \wedge *\bar{g}$, where f, g are both differential forms of the same degree. It is easy to prove that formula 1 is compatible with respect to the Hodge star metric.

The following proposition describes the relation between the adjoints of the usual operators with respect to the Hodge star metric and the operators themselves.

Proposition 27 *We have that $d^* = -*d*$. This implies that $\Delta = dd^* + d^*d$ commutes with $*$.*

Proof: Let f, g be k-forms. Recall that $** = w$ and $*w* = 1$. By Stokes' theorem we have

$$\begin{aligned}\langle df, g\rangle &= \int_X df \wedge \bar{*}g = \int_X d(f \wedge \bar{*}g) + (-1)^k \int_X f \wedge d\bar{*}g \\ &= (-1)^k \int_X f \wedge d\bar{*}g = (-1)^k \int_X f \wedge \bar{*}\bar{*}wd\bar{*}g \\ &= -\int_X f \wedge \bar{*}(\bar{*}d\bar{*})g = -\langle f, \bar{*}d\bar{*}g\rangle\end{aligned}$$

I used that if g is a k form, then $\bar{*}g$ is a $(2n-k)$ form, $d\bar{*}g$ is a $(2n-k+1)$ form. Since by definition $w = \sum(-1)^i p_i$ and so w acts on this form as multiplication by $(-1)^{k-1}$.

The commutation relation follows from the adjointness relation.

■

The operator $*$ may be extended to a vector bundle $\wedge T^*X \otimes \mathcal{E}$. This way we get an operator $\bar{*}_E$ from the space of (p,q)-forms on $\mathcal{E}$ to the space of $(2n-p, 2n-q)$-forms on $\mathcal{E}^*$. The following proposition is proven in a similar way to the previous one.

Proposition 28 *We have $\bar{\partial}^* = -\bar{*}_E\bar{\partial}\bar{*}_E$. Therefore the complex Laplacian $\overline{\Box} = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ commutes with the bundle star operator.*

4.3 Representations of $\mathfrak{sl}_2(\mathbb{C})$

This section assumes familiarity with Lie groups, Lie algebras, representations of Lie algebras.

There are elements $E_+, E_-, H \in \mathfrak{sl}_2(\mathbb{C})$ so that $[H, E_+] = 2E_+, [H, E_-] = -2E_-, [E_+, E_-] = H$.

For a representation π of $\mathfrak{sl}_2(\mathbb{C})$ on a finite dimensional vector space V (recall that by Weyl's unitary trick this is completely reducible) we have the following actions on weight spaces:

$$\begin{aligned}\pi(E_+) &: V^\lambda \longrightarrow V^{\lambda+2} \\ \pi(E_-) &: V^\lambda \longrightarrow V^{\lambda-2}\end{aligned}$$

There is one irreducible representation (up to equivalence) of $\mathfrak{sl}_2(\mathbb{C})$ on V. If $\{v_0, \dots, v_d\}$ is a basis of V then the representation is given by $\pi(H)v_k = (d-2k)v_k, \pi(E_+)v_k = (d+1-k)v_{k-1}, \pi(E_-)v_k = (k+1)v_{k+1}$.

Let E be a hermitian vector space with the operators $L = \Omega\wedge$ and L^* on $\wedge E^*$. Consider the representation π of $\mathfrak{sl}_2(\mathbb{C})$ on $\wedge E^*$ given by $\pi(E_+) = L^*, \pi(E_-) = L, \pi(H) = [L^*, L] = \sum_{k=0}^{2n}(n-k)p_k$. This is a representation since the commutation relations hold, as proven in the previous section.

Let π' be the restriction of π to SU_2 (note that π' is unitary).

For a k-form f, let $\tilde{f}$ also denote the function $f\wedge$. For example $L = \tilde{\Omega}$ and $L^* = \tilde{\Omega}^*$.

Proposition 29 *For any k-form f we have*

$$\pi'(w)\tilde{f}\pi'(w)^{-1} = -iJ\tilde{f}^*J^{-1}.$$

Proof: The proof of this proposition will use a differential equation for a more general function $\tilde{f}_x$. Define $\tilde{f}_x = \pi'(2xw/\pi)\tilde{f}\pi'(2xw/\pi)^{-1}$.

Then $\tilde{f}_{\pi/2}$ is the left hand side of the proposition.

It is easy to show that $[L^*, \tilde{f}] = -J\tilde{f}^*J^{-1}$. Since $L^* = \tilde{\Omega}^*$ commutes with both J and $\tilde{f}^*$ we get that $[L^*, [L^*, \tilde{f}]] = 0$. Therefore using the power series expansion of exp we get the differential equation for $\tilde{f}_x$:

$$\begin{aligned}\tilde{f}'_x &= i(\mathrm{ad}(L^*) + \mathrm{ad}(L))\tilde{f}_x \\ \tilde{f}_0 &= \tilde{f}\end{aligned}$$

The unique solution to this equation is $\tilde{f}_x = \cos x\tilde{f} + i\sin x\mathrm{ad}(L^*)\tilde{f}$.

The result now follows from the fact that $\tilde{f}_{\pi/2} = i[L^*, \tilde{f}] = -iJ\tilde{f}^*J^{-1}$.

■

Having made the connection between our $\mathfrak{sl}_2(\mathbb{C})$ representation and the operators L, L^*, we still need a connection with the Hodge star operator. The following proposition will give this connection.

Proposition 30 *If f is a k-form, then $*f = i^{k^2-n}J^{-1}\pi'(w)f$.*

Proof: For k-forms we have that $*\tilde{f} = (-1)^k\tilde{f}^**$ and $*1 = v$, the volume form. Define the operator $\tilde{*} = i^{k^2-n}J^{-1}\pi'(w)$. In order to prove the equality, it is enough to prove that this operator satisfies the same two conditions as $*$.

The rest of the proof follows from the previous proposition (for the second condition) as well as the general form of the representation π (for the first condition). ■

Define the commutating operator $d_c = J^{-1}dJ = wJdJ$. On functions this is equivalent to $d_c = -i(\partial - \bar{\partial})$ and so $dd_c = 2i\partial\bar{\partial}$.

4.4 The Kähler Condition

Let X be a hermitian complex manifold and let Ω be the associated $(1,1)$-form.

Definition 31 *The manifold is called* **Kähler** *is $d\Omega = 0$.*

Lemma 32 *If X is Kähler then $[L, d] = 0$ and $[L, d^*] = d_c$.*

Proof: $[L, d]f = Ldf - dLf = \Omega \wedge df - d(\Omega \wedge f) = \Omega \wedge df - \Omega \wedge df = 0$.

We will prove the adjoint of the second relation. Let f be a k-form.

Just like in the proof of Proposition 29 we take $d_x = \pi'(2xw/\pi)d\pi'(2xw/\pi)^{-1}$. Similarly to the proof there we get that $d_x = \sum \frac{1}{k!}\text{ad}^k(ix(L + L^*))d$. The Kähler condition translates as $\text{ad}(L)d = 0$.

This implies that $d_x = \sum a_k(x)\text{ad}^k(L^*)d$. So $d_{\pi/2} = \lambda \text{ad}(L^*)d$.

But by definition we get that this is equal to $\pi'(w)d\pi'(w)^{-1}$. Lemma 32 implies that $\pi'(w)f = i^{n-k^2}J * f$. Also $\pi'(w)^{-1}f = i^{k^2-n} *^{-1} J^{-1}f$. Therefore we get that $\pi'(w)d\pi'(w)^{-1}f = iJd^*J^{-1}f$.

Therefore, we get that $\text{ad}(L^*)d$ divides iJd^*J^{-1}. Therefore $\text{ad}^k(L^*)d = 0$ for $k \geq 2$. So $d_x = a_0(x)d + a_1(x)\text{ad}(L^*)d$.

As before we get that $d_x = \cos xd + i\sin x\text{ad}(L^*)d$. Therefore $d_{\pi/2} = i[L^*, d] = iJd^*J^{-1} = -iJ^{-1}dJ = -id_c$. ■

There are operators $\Delta, \Box, \overline{\Box}$ on X. In general there is no efficient way to relate these operators, but in the case of Kähler manifold, the fact that the associated form is closed yields the following theorem:

Theorem 33 *If X is Kähler then*

$$\Delta = 2\Box = 2\overline{\Box}.$$

Proof: We have that $[\Delta, L] = -d[L, d^*] - [L, d^*]d$. Therefore, the previous Lemma implies that $[\Delta, L] = -dd_c - d_cd$. But we saw that $dd_c + d_cd = 2i(\partial\bar{\partial} + \bar{\partial}\partial) = 0$. Therefore $L\Delta = \Delta L$.

Note that $\Delta = dd^* + d^*d = d[L^*, d_c] + [L^*, d_c]d$, a dual of the previous Lemma. Therefore $4\Box = 4(\partial\partial^* + \partial^*\partial) = \Delta + \Delta_c = 2\Delta$. The conjugate relation is proven similarly. ■

5 The Hodge Decomposition Theorem

We now have the tools to prove the Hodge decomposition theorem for compact Kähler manifolds.

Consider the de Rham complex $\wedge^*T^*X(X)$ and the Dolbeault complex $\wedge^{p,*}T^*X(X)$. We calculated the symbols of the operators d and $\bar{\partial}$ in Section 3.1. These prove that the de Rham and Deolbeault complexes are actually elliptic.

Therefore, Theorem 24 implies that we have

$$\mathcal{K}_k(X) = \mathcal{K}(\wedge^kT^*X(X)) = H^k(X, \wedge^*T^*X(X)) = H^k_{dR}(X) = H^k(X, \mathbb{C}).$$

Moreover, for the Dolbeault complex we have that

$$\mathcal{K}_{p,q}(X) = \mathcal{K}(\wedge^{p,q}T^*X(X)) = H^q(X, \wedge^{p,*}T^*X(X)) = H^{p,q}(X) = H^q(X, \Omega^p),$$

this last equality coming from the Dolbeault lemma.

Theorem 34 (Hodge) *Let X be a compact Kähler manifold. Then*

$$H^n(X, \mathbb{C}) = \oplus_{p+q=n} H^{p,q}(X),$$

with $\bar{H}^{p,q} = H^{q,p}$.

Proof: By the previous discussion, it is enough to prove that the same relation holds on the level of harmonic forms.

Let $f \in \mathcal{K}_n$. Then $\Delta f = 0$. By Theorem 33 we get that $\overline{\Box} f = 0$. Writing out the homogeneous terms $f = \sum f_{p,q}$ we get that $\sum \overline{\Box} f_{p,q} = 0$.

Note that $\overline{\Box}$ preserves each of the spaces $\wedge^{p,q}T^*X(X)$ and so each $\overline{\Box} f_{p,q} = 0$. Therefore we get a map $\mathcal{K}_n \longrightarrow \oplus \mathcal{K}_{p,q}$.

Bijectivity follows from construction.

The conjugation property is a simple consequence of the conjugation property on the level of global sections of the bigraded cotangent space. ■

6 Applications

6.1 Duality Theorems

The following duality theorem is useful for proving Riemann-Roch formulae.

Theorem 35 (Serre, Kodaira) *Let X be a compact complex manifold of complex dimension n. Let E be a holomorphic vector bundle on X. Then*

$$H^q(X, \Omega^p(E)) \xrightarrow{\simeq} H^{n-q}(X, \Omega^{n-p}(E^*)).$$

Proof: Follows from the Dolbeault lemma and the fact that the operator $\bar{*}_E$ is an isomorphism. ■

6.2 Topological Invariants

Define the Hodge numbers $h_{p,q} = \dim_{\mathbb{C}} H^q(X, \Omega^p) = \dim \mathcal{K}^{p,q}(X)$. The Dolbeault lemma guarantees that these numbers are finite. These are topological invariants, even though apparently depending on the hermitian structure on X.

Let E be a holomorphic vector bundle on X. Define $h_{p,q}(E) = \dim_{\mathbb{C}} H^q(X, \Omega^p(E))$. Define the Euler characteristic as $\chi(E) = \sum (-1)^k h_{0,k}$.

Theorem 36 (Hirzebruch-Riemann-Roch) *If X is a compact complex manifold and E is a holomorphic vector bundle then we have*

$$\chi(E) = \langle ch(E)Td(X), [X] \rangle,$$

where ch is the Chern character and Td is the Todd genus.

A proof of this theorem may be found in [?]. I will show that it reduces to the classical Riemann-Roch theorem when X is a compact Riemann surface and E is a holomorphic line bundle.

Theorem 37 (Riemann-Roch) *Let X be a compact Riemann surface and D a divisor on it. Let $\Omega(D)$ be the sheaf of divisors which are $\geq -D$ at each point. If $l(D) = \dim H^0(X, \Omega(D))$ and K is the canonical divisor, then*

$$l(D) - l(K - D) = \dim D + 1 - g.$$

Proof:

Let $E = \Omega(D)$, a rank 1 vector bundle. Then by Hirzebruch-Riemann-Roch we get that $\chi(E) = \langle \mathrm{ch}(E)Td(X), [X]\rangle$.

We have that $\chi(E) = h^{0,0} - h^{0,1} = \dim H^0(X, \Omega(D)) - \dim H^1(X, \Omega(D)^*)$. By Serre duality this is equal to $\dim H^0(X, \Omega(D)) - \dim H^0(X, \Omega(K-D)) = l(D) - l(K-D)$.

Therefore we only need to prove that $\langle \mathrm{ch}(E)Td(X), [X]\rangle$. But $ch(E) = 1 + c_1(E)$ and $Td(X) = 1 + c_1(X)/2$. Therefore $\langle \mathrm{ch}(E)Td(X), [X]\rangle = \langle (1 + c_1(E))(1 + c_1(X)/2), [X]\rangle = \langle c_1(E) + c_1(X)/2, [X]\rangle = \chi(X)/2 + c_1(E)[X] = 1 - g + \deg D$.

To see that $c_1(E)[X] = \deg D$, we need that $c_1(E) = \deg D \cdot h$, where h is the generator of $H^2(X)$. Note that if D is just one point, then $\Omega(D)$ is the canonical line bundle. Moreover, $\Omega(D + D') = \Omega(D) \otimes \Omega(D')$. Since c_1 is additive with respect to $\otimes$, we get the desired result.

■

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