Eilenberg-MacLane Spaces and Cohomology

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Abstract

The Eilenberg-MacLane spaces are spaces $K(G, n)$ with the property that $\pi_n(K(G, n)) = G$ and all other homotopy groups are trivial. Their simplicity makes them very useful in the classification of general topological spaces. The first part of the paper deals with general algebraic topology. The second part of the paper proves the existence of Eilenberg-MacLane spaces and gives a proof of their uniqueness. The last part of the paper proves that the group cohomology of a group G is the same as the singular cohomology with local coefficients of the Eilenberg-MacLane space $K(G, 1)$.

1 Introduction

1.1 Equivariant Cohomology

1.1.1 Definitions

Let X be a topological space, and G a group acting on X on the left. Let R be an abelian group on which G acts.

Define the cochain complexes in the standard way.

Definition 1 *1. $C_e^q(X,R) = \{f \in C^q(C,R) | gf(T) = f(gT), \forall T \in C_q(X,R)\}$ (gT represents the action of G on X, while $gf(T)$ represents the action on R) is called the group of equivariant cochains.*

2. $C_r^q(X,R) = \{f \in C^q(X,R) | \delta f \in C_e^{q+1}(X,R)\}$ is called the group of residual cochains.

Note that f equivariant implies that δf is also equivariant, and similarly for residual. Define $H_e^q(X,R)$ as the cohomology group associated with the complex $C_e^*(X,R)$ and $H_r^q(X,R)$ as the cohomology group of the complex $C_r^*(X,R)$.

The exact sequence $0 \longrightarrow C_e^*(X,R) \xrightarrow{i} C^*(X,R) \xrightarrow{j} C_r^*(X,R) \longrightarrow 0$ yields the following long exact sequence on cohomologies, in the standard way:

Proposition 1

$$\ldots \longrightarrow H_e^q(X,R) \xrightarrow{i^*} H^q(X,R) \xrightarrow{j^*} H_r^q(X,R) \xrightarrow{\delta} H_e^{q+1}(X,R) \longrightarrow \ldots$$

1.1.2 Equivariant Chain Maps

Given two chain complexes A and B, on which the group G acts on the left (such as $A = C_*(X)$), a chain map is a map $f : A \longrightarrow B$ that commutes with the boundary operators.

Note that any chain map which is equivariant with respect to the action of G induces (vertical) homomorphisms that make the following diagram commutative:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & H_e^q(A,R) & \xrightarrow{i^*} & H^q(A,R) & \xrightarrow{j^*} & H_r^q(A,R) & \xrightarrow{\delta} & H_e^{q+1}(A,R) & \longrightarrow & \cdots \\ & & \downarrow f_e^* & & \downarrow f^* & & \downarrow f_r^* & & \downarrow f_e^* & & \\ \cdots & \longrightarrow & H_e^q(B,R) & \xrightarrow{i^*} & H^q(B,R) & \xrightarrow{j^*} & H_r^q(B,R) & \xrightarrow{\delta} & H_e^{q+1}(B,R) & \longrightarrow & \cdots \end{array}$$

As in general, we have the notion of equivariant chain-homotopy:

Definition 2 *Given equivariant chain maps f, g from a complex A to itself, we say f and g are equivariant chain-homotopic if there is an equivariant map Δ so that $f - g = \partial\Delta - \Delta\partial$.*

1.2 Covering Spaces

Let X be a topological space and let $PX = \{f : [0,1] \longrightarrow X | f(0) = x_0\}$. Define the equivalence relation $f \sim g$ if the loop $f \cdot g^{-1}$ is nulhomotopic. Then the universal covering space of X is $\tilde{X} = PX/\sim$.

The following are a few known remarks about the universal covering space:

Remark 1 *1. $\tilde{X}$ is simply connected.*

2. The projection map $e : PX \longrightarrow X$ defined by $e(f) = f(1)$ lifts any singular complex from X to $\tilde{X}$.

3. $\pi_1(X)$ acts on $\tilde{X}$ freely on the left.

One can generalize the notion of universal covering space to that of covering space. Let G be a normal subgroup of $\pi_1(X)$. Then define $f \sim_G g$ if $f \cdot g^{-1} \in G$. The covering space (associated with the group G) $\tilde{X}_G = PX/\sim_g$ will have the property that $\pi_1(\tilde{X}_G) = G$. Moreover, the group $\pi_1(X)/G$ acts on $\tilde{X}_G$.

1.3 Homotopy Theory

This section contains several definitions and propositions used in this paper.

Definition 3 *A topological space X is called n-connected if $\pi_i(X, x_0) = 0$ for all $i \leq n$. A pair (X, A) is called n-connected if $\pi_i(X, A, x_0) = 0$ for all $i \leq n$.*

Proposition 2 *A CW-complex pair (X, A) is r-connected so that A is s-connected. Then for all $i \leq r+s$ the quotient map $X \longrightarrow X/A$ induces an isomorphism $\pi_i(X, A, x_0) \longrightarrow \pi_i(X/A)$. If $i = r+s+1$ then the induced map is a surjection.*

Proof: Attach the cone of A (denoted by $C(A)$ along A to the space X. Then $C(A)$ is a contractible subcomplex of $X \cup C(A)$ by construction. Therefore the map $X \cup C(A) \longrightarrow (X \cup C(A))/C(A) = X/A$ is a homotopy equivalence. Since $C(A)$ is contractible the LES of pair gives the isomorphism $\pi_i(X \cup C(A)) = \pi_i(X \cup C(A), C(A))$ so we have maps

$$\pi_i(X, A) \longrightarrow \pi_i(X \cup C(A)) = \pi_i(X \cup C(A), C(A)) \longrightarrow \pi_i((X \cup C(A))/C(A)) \longrightarrow \pi_i(X/A).$$

Since $(C(A), A)$ is clearly $(s+1)$-connected, from the LES for the pair. Therefore we may apply homotopy excision for the pairs (X, A) and $(X \cup C(A), C(A))$ and get the required isomorphisms. ∎

Proposition 3 *Let $n \geq 2$ and let F be a free group on generators $\alpha \in \mathcal{A}$. Then $\pi_i(\bigvee_{\alpha \in \mathcal{A}})) = F$ if $i = n$ and 0 if $i < n$.*

Proof: First, assume that there are finitely many elements in F. Then $\bigvee_\alpha S^n$ is the n-skeleton of the CW-complex product $\prod_\alpha S^n$. The CW-complex structure of this product has cells in dimension $0, n, 2n, \ldots$ so this means that the pair $(\prod_\alpha S^n, \bigvee_\alpha S^n)$ is $(2n-1)$-connected. This means that the LES for the pair $(\prod_\alpha S^n, \bigvee_\alpha S^n)$ gives isomorphisms for $\pi_i(\prod_\alpha S^n) = \pi_i(\bigvee_\alpha S^n)$ for $i < 2n-1$ and in particular for $i = n \geq 2$.

But we know that $\pi_n(\prod_\alpha S^n) = \oplus_\alpha \pi_n(S^n)$ is a free abelian group so we are done in the finite case.

Drop the finiteness assumption on the number of generators. Then we get a natural inclusion homomorphism from $\oplus_\alpha \pi_n(S^n) \longrightarrow \pi_n(\bigvee_\alpha S^n)$. Since the image of any loop in $\bigvee_\alpha S^n$ is compact, hence in the wedge of a finite number of spheres, surjectivity of the homomorphism in the finite case implies the surjectivity of the homomorphism in the infinite case. Moreover, a nulhomotopy has compact image, which means that it is contained in a finite wedge of spheres. Therefore any zero element in $\pi_n(\bigvee S^n)$ will come from an element in the direct sum of a finite number of the $\pi_n(S^n)$. Therefore the injectivity in the finite case implies injectivity in the infinite case.

Clearly the wedge of circles is $(n-1)$-connected by Hurewicz's theorem for example. (Or by default if $n = 1$.) ■

2 Eilenberg-MacLane Spaces

Definition 4 *Let G be a group. An Eilenberg-MacLane space, denoted by $K(G,n)$, is a topological space so that $\pi_n(K(G,n)) = G$ and $\pi_k(K(G,n)) = 0$ if $k \neq n$.*

Remark 2 *An alternative definition of $K(G,n)$ is a topological space with $\pi_k(K(G,n)) = 0$ and contractible universal cover. Whitehead's theorem implies that the two definitions are equivalent.*

2.1 Existence Theorems

2.1.1 $K(G,1)$

Theorem 4 *Let G be any group. Then there exists a CW-complex $K(G,1)$.*

Proof: Consider a free presentation of the group

$$0 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 0,$$

where F is free generated by $\alpha \in \mathcal{A}$ and R, normal in F, is the set of relations generated by $\beta \in \mathcal{B}$.

The first step is to construct a CW complex of $\pi_1 = G$. Let $X^1 = \bigvee_{\alpha \in \mathcal{A}} S^1$, with x_0 be the common point of the circles. Then by the van Kampen theorem, $\pi_1(X^1) = F$. We will inductively construct the n-skeleton X^n of a CW-complex X with $\pi_1(X^n) = G$ and $\pi_k(X^n) = 0$ for $k > 1$. Then we will clearly have that $X = \bigcup X^n = K(G,1)$.

The main tool of the induction is the theorem that for a CW-complex the n-th homotopy group is determined by the $(n+1)$-th skeleton. Therefore the base case is the construction of X^2 so that $\pi_1(X^2) = G$. Once we have done that, we no longer need to concentrate upon π_1.

Let $\hat{\beta} : (S^1, 1) \longrightarrow (X^1, x_0)$ be the image of $\beta \in \mathcal{B}$ in $F = \pi_1(X^1, x_0)$. Define $X^2 = X^1 \bigcup (\cup_{\hat{\beta}} e^2_\beta)$, where $\cup e^2_\beta$ means that it is attached using the attaching map $\hat{\beta}$.

The cellular approximation theorem yields that $\pi_1(X^2, X^1, x_0) = 0$ (since any map from $(I, \partial I, J^) \longrightarrow (X^2, X^1, x_0)$ is homotopic to a cellular map which is zero. Therefore the long exact sequence (LES) for the pair (X^1, X^2) will be

$$\ldots \longrightarrow \pi_2(X^2, X^1, x_0) \longrightarrow F = \pi_1(X^1, x_0) \xrightarrow{i_*} \pi_1(X^2, x_0) \longrightarrow 0.$$

For any generator β of R there is an obvious element in $\pi_2(X^2, X^1, x_0)$ (just the map that goes to the e^2_β cell which is $\hat{\beta}$ on the boundary) that maps to $\beta \in \pi_1(X^1, x_0)$. This means that $R \subset \ker i_*$.

Let $\tilde{X}^1$ be the covering space of X^1 associated with the (normal) group $R \subset F$. Then $\pi_1(\tilde{X}^1) = R$ and there is a fibration $F/R \longrightarrow \tilde{X}^1 \longrightarrow X^1$. There is a natural extension of $\tilde{X}^1$ to $\tilde{X}^2$ that creates a covering space $\tilde{X}^2$ of X^2 with fiber F/R as well.

Then the fibration long exact sequences, together with the long exact sequences for pairs (X^2, X^1) and $(\tilde{X}^2, \tilde{X}^1)$ give the following commutative diagram:

$$\begin{array}{ccccc}
R = \pi_1(\tilde{X}^1, x_0) & \longrightarrow & F = \pi_1(X^1, x_0) & & \\
\downarrow u & & \downarrow i_* & & \\
\pi_1(\tilde{X}^2, x_0) & \xrightarrow{v} & \pi_1(X^2, x_0) & \longrightarrow & F/R(\longrightarrow 0) \\
\downarrow & & \downarrow & & \\
0 & & 0 & &
\end{array}$$

Then the image of R in F is in the kernel of i_* as before, since the image is generated by the generators β. Therefore the commutativity of the diagram implies that $vu = 0$. But the exact sequence implies that u is surjective and that means that $v = 0$. Therefore we have $\pi_1(X^2, x_0) = F/R = G$.

Let's proceed by induction. Assume that we have constructed X^m so that $\pi_1(X^m, x_0) = G$ and for $1 < i < m$ we have $\pi_i(X^m, x_0) = 0$. Then choose generators $\gamma : (S^m, 1) \longrightarrow (X^m, x_0)$ of $\pi_m(X^m, x_0)$. As before, construct

$$X^{m+1} = X^m \bigcup (\cup_\gamma e^{m+1}_\gamma),$$

using the γ as attacging maps. By construction, $\pi_{m+1}(X^{m+1}, X^m, x_0) = \bigoplus_\gamma \mathbb{Z}$ because we may choose the generators to be precisely those that define the attached $(m+1)$-cells. This means that the map $\pi_{m+1}(X^{m+1}, X^m, x_0) \longrightarrow \pi_m(X^m, x_0)$ is surjective and as before (need here that $\pi_m(X^{m+1}, X^m, x_0) = 0$), the LES for the pair (X^{m+1}, X^m) proves that $\pi_m(X^{m+1}, x_0) = 0$. Since all smaller homotopy groups are determined by X^m, they remain the same. Now $X = \bigcup X^m$ will be the sought $K(G, 1)$.

■

2.1.2 $K(G, n)$ for $n \geq 2$

Theorem 5 *Let G be any abelian group and let $n \geq 2$ be an integer. Then there exists a CW-complex $K(G, n)$.*

Proof: Start as before with a free presentation

$$0 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 0,$$

with F free abelian generated by $\alpha \in \mathcal{A}$ and R a subgroup generated by $\beta \in \mathcal{B}$. Define $X^n = \bigvee_\alpha S^n$.

Define $X^{n+1} = X^n \bigcup(\cup_\beta e_\beta^{n+1})$. Finding the homology of X^{n+1} is trivial and by Hurewicz's theorem we get that the space X^{n+1} is $(n-1)$-connected and has G as its n-th homotopy group. $(n \geq 2.)$

Now an inductive definition of X^m as for $K(G, 1)$ will create the m-skeleta of a CW-complex $X = \bigcup X^m = K(G, n)$.

■

2.2 Main Property

The main property of the Eilenberg-MacLane spaces is the following

Theorem 6 *Let Y be a CW-complex. Then there is a bijection*

$$[Y, K(G, n)] = H^n(Y, G),$$

where $[X, Y]$ represents the homotopy classes of maps.

The proof of this, which I omit, can be done using general cohomology theories ([Hatcher]) or obstruction theory ([Mosher, Tangora]).

2.3 Uniqueness

Uniqueness is characterized by the following property:

Theorem 7 *The homotopy type of $K(G, 1)$ is uniquely determined by G.*

Proof: Since $K(G, n)$ is a CW-complex, we may apply theorem 5. Then we have

$$[K(G, n), K(G, n)] = H^n(K(G, n), \pi_n(K(G, n))) = H^n(K(G, n), G).$$

Since all the cells of $K(G, n)$ have dimension at least n in the CW-complex, $H_{n-1} = 0$ so by the universal coefficient theorem

$$[K(G, n), K(G, n)] = \mathrm{Hom}(H_n(K(G, n)), G) = \mathrm{Hom}(G, G),$$

where the last equality if given by Hurewicz's theorem.

Now because of this equality, we may lift the identity on G to a function on $K(G, n)$. But then, this function induces the identity, i.e., an isomorphism on $\pi_n(K(G, n))$, and is 0 on all other homotopy groups (although still an isomorphism there, since the homotopy groups are 0). Then by Whitehead's theorem, this function induces a homotopy equivalence between the two $K(G, n)$ spaces. ■

2.4 Examples

The constructive proof given in section 2.1 produces some very large and counterintuitive CW-complexes. This section is concerned with the actual construction of Eilenberg-MacLane spaces.

Example 1 *$S^1 = K(\mathbb{Z}, 1)$, a well-known theorem in algebraic topology.*

Proposition 8 *$K(G \times H, 1) = K(G, 1) \times K(H, 1)$.*

Proof: Follows from the fact that for all k we have $\pi_k(X \times Y) = \pi_k(X) \oplus \pi_k(Y)$. ■

Example 2 *If $\Omega K(G, n) = K(G, n-1)$ (where Ω represents the loop space).*

Proof: It is known that $e : PX \longrightarrow X$ previously defined is a Serre fibration with PX contractible. Then the long exact sequence of homotopy groups for fibrations gives isomorphisms $0 \longrightarrow \pi_n(X, x_0) = \pi_{n-1}(\Omega X, f_0)$. This proves the example. ■

Example 3 *The infinite lens space $L_m = S^\infty/(\mathbb{Z}/m)$ (under the action $\vec{z} \mapsto e^{\frac{2\pi i}{m}}\vec{z}$) is a $K(\mathbb{Z}/m, 1)$ space.*

Proof:

Since $S^\infty = \lim S^n$ (direct limit), and for large enough n we have $\pi_k(S^n) = 0$ it means that S^∞ is contractible.

Since S^∞ is the universal cover of L_m, it is enough to show that $\pi_1(L_m) = \mathbb{Z}/m$. The exact sequence for the fibration $\mathbb{Z}/m \longrightarrow S^\infty \longrightarrow L_m$ gives an injective map $\pi_1(L_m, x_0) \longrightarrow \pi_0(\mathbb{Z}/m, f_0)$. Moreover, any path from $\vec{z}$ to $e^{\frac{2\pi i}{m}}\vec{z}$ on S^∞ is a loop in L_m. Since S^∞ is simply connected, it means that a multiple of m of this loop is zero and also the order of the loop is m. Therefore, $\pi_1(L_m) = \pi_0(\mathbb{Z}/m)$ (by element count) and is generated by an element of order m. So $\pi_1(L_m) = \mathbb{Z}/m$. ■

Remark 3 *These examples allow us to construct natural $K(G, 1)$ spaces for any finitely generated abelian group G, using the previous proposition.*

A more nontrivial construction of a $K(G, n)$ space is offered by the following theorem, whose proof can be found in ([Hatcher], 484):

Theorem 9 *(Dold-Thom) The functor $X \mapsto \pi_i(SP(X))$ coincides with the functor $X \mapsto H_i(X)$, where $SP(X)$ represents the infinite symmetric product of X.*

Example 4 *Since $H_i(S^n) = \mathbb{Z}$ if $i = n$ and 0 for $i > 0, i \neq n$, this means that the Dold-Thom theorem implies that $SP(S^n) = K(\mathbb{Z}, n)$.*

More importantly, this theorem provides an alternative construction for Eilenberg-MacLane spaces for $n \geq 2$.

Definition 5 *Let G be an abelian group. An $M(G,n)$ Moore space is a CW-complex X so that $H_n(X) = G$ and $\tilde{H}_i(X) = 0$ otherwise.*

Clearly the Dold-Thom theorem proves that $SP(M(G,n)) = K(G,n)$. Therefore the existence of Eilenberg-MacLane spaces comes down to the existence of Moore spaces. The construction ([Hatcher], 143) is analogous to our proof, starting with a wedge of spheres for the free presentation of G, but stops after one step of adding $(n+1)$-cells to the complex.

3 Group Cohomology

Let G be a group. There is a natural way of defining an associated abstract complex K_G by letting a k-cell in K_G be any $k+1$-tuple $(g_0, \ldots, g_k) \in G^{k+1}$. Define the boundary map in the standard way and let G act on K_G by setting $g(g_0, \ldots, g_k) = (gg_0, \ldots, gg_k)$.

Proposition 10 $\tilde{H}^i(K_G, R) = 0$.

Proof: Define the operator $\Delta : C_q(K_G, R) \longrightarrow C_{q+1}(K_G, R)$ by letting $\Delta(g_0, \ldots, g_k) = (1, g_0, \ldots, g_k)$. Then it is easy to see that $\partial\Delta + \Delta\partial = 1$. Define Δ^* in the standard way on $C^{k+1}(K_G, R)$. We have $\Delta^*\delta + \delta\Delta^* = 1$ and the result follows from this. ∎

This proposition breaks up the LES for equivariant and residual cohomology into isomorphisms $H^k_e(K_G, R) = H^{q-1}_r(K_G, R)$. This means that the only sensible cohomology is the equivariant one.

Definition 6 *Consider a group G acting on the right on a topological abelian group R. Then define $H^k(G, R) = H^k_e(K_G, R)$.*

4 Cohomology with Coefficients in Group Bundles

Consider X a topological space and $\mathcal{R}$ a group bundle on X with fiber R, an abelian group. (A more detailed definition would be an assignment $X \ni x \mapsto R_x$ an abelian group such that any class of paths from x to y defines an isomorphism $\psi_{xy} : R_x \longrightarrow R_y$ in a coherent manner, i.e., $\psi_{yz}\psi_{xy} = \psi_{xz}$.)

Given a base-point x_0 with fiber $R = R_{x_0}$, the group bundle defines a natural right action of $\pi_1(X)$ on R, $g \mapsto g\sigma$. It also gives a natural left action $g \mapsto \sigma g = g\sigma^{-1}$.

The group bundle allows us to define cohomology with local coefficients in the fibers at each point.

For a simplicial complex $T : \Delta_q \longrightarrow X$ define $R_T = R_{T(e_0)}$. If $T^{(i)} = T|_{[e_0, \ldots, \hat{e}_i, \ldots, e_q]}$ then $R_{T^{(i)}} = R_T$ for $i \neq 0$. In order to define the boundary map on singular complexes we need to define coefficients for each face, so we need a correlation map from $R_{T^{(0)}}$ to R. Choose $\rho = T|_{[e_0, e_1]}$.

4.1 Homology

Define $C_q(X, \mathcal{R}) = \{\sum g_i T_i | g_i \in R_{T_i}\}$ and define $\partial : C_q \longrightarrow C_{q-1}$ be setting $\partial(gT) = g\rho T^{(0)} + \sum_{i=1}^{q}(-1)^i gT^{(i)}$, the obvious formula. It is easy to see that ∂ is a boundary map and that it defines homology groups.

4.2 Cohomology

Define $C^q(X, \mathcal{R}) = \langle \mathrm{Hom}(T, R_T) \rangle$. The coboundary operator is similarly defined

$$(\delta f)(T) = \rho f(T^{(0)}) + \sum_{i=1}^{q+1} (-1)^i f(T^{(i)}).$$

Again, δ is a valid coboundary map and it defines cohomology groups $H^q(X, \mathcal{R})$.

4.3 Relation to Equivariant Cohomology

Theorem 11 *Let X be a topological space with an associated group bundle $\mathcal{R}$. Let $\tilde{X}$ be the universal covering space of X. For the natural left action of $\pi_1(X)$ on $R = R_{x_0}$ we have an isomorphism*

$$H^q(X, \mathcal{R}) \xrightarrow{\sim} H^q_e(\tilde{X}, R).$$

Proof: Let $x \in \tilde{X}$. Then x is a class of paths from x_0 to $e(x)$ (see 1.2) and $\pi_1(X)$ acts on $\tilde{X}$ in the obvious way and so $x(ag) = (xa)g$ for any $a \in \pi_1(X)$.

Define the function $E : C^q(\tilde{X}, R) \longrightarrow C^q(X, \mathcal{R})$ by setting $Ef(T) = xf(eT)$, where $x = T(e_0)$. Now $f(eT) \in R_{eT} = R_{e(x)} \Longrightarrow Ef(T) \in R_T$.

Proposition 12

$$\delta(Ef) = E\delta f.$$

Proof:

$$\delta(Ef)(T) - E\delta f(T) = \sum(-1)^i x^{(i)} f(eT^{(i)}) - \sum(-1)^i x f(eT^{(i)}) - x\rho f(eT^{(0)}) = 0,$$

for trivial reasons. ■

Proposition 13 $\ker e = 0$.

Proof: Assume that $f \neq 0$ so there is some singular simplex T so that $f(T) \neq 0$. Choose $\tilde{T}$ so that $e\tilde{T} = T$, in which case we have $ef(\tilde{T}) = \tilde{T}(e_0)f(T) \neq 0$. ■

Proposition 14 *$Ime = C^q_e(\tilde{X}, R)$.*

Proof: Let $a \in \pi_1(X)$. Then $ef(aT) = aT(e_0)f(eaT) = a(T(e_0)f(eT)) = aef(T)$. This proves that the map is well-defined into the set of equivariant cochains.

Define $e^{-1}(h)(T) = x^{-1}h(\tilde{T})$, where $x = \tilde{T}(e_0)$ and $e\tilde{T} = T$. Since h is equivariant, the above formula is $\pi_1(X)$ invariant in terms of $\tilde{T}$, which means that it is independent of the lifting. Therefore surjection follows. ■

These three propositions prove the theorem clearly.

5 Cohomology of $K(G, 1)$

Let $\mathcal{R}$ be an abelian group bundle on $K(G, 1)$. Then $G = \pi_1(K(G, 1))$ acts on $R = R_{x_0}$ and we have the following

Theorem 15

$$H^i(K(G, 1), \mathcal{R}) = H^i(G, R),$$

where the first one represents cohomology with coefficients in the group bundle and the second one is group cohomology.

Proof: Let $X = K(G, 1)$ and let $\tilde{X}$ be the universal covering space. By definition, $\tilde{X}$ is contractible so $\pi_i(\tilde{X}) = 0$ for $i \geq 1$.

By Hurewicz's theorem, we have that $H_k(K(G, 1), \mathbb{Z}) = 0$ for all k and the universal coefficient theorem for homology gives that $H^k(K(G, 1), \mathbb{Z}) = 0$.

In light of the previous two theorems, all we need to do is prove is the main theorem of the following subsection:

5.1 Relation between Group and Singular Cohomologies

For the action of G on $\tilde{X}$, consider representatives for all the orbits. Since $\pi_1(X) = G$ acts freely on $\tilde{X}$ the orbits are disjoint. Therefore, for each $x \in \tilde{X}$, we associate $\pi_x \in G$ so that $x = \pi_x \hat{x}$, where $\hat{x}$ is a (fixed) representative of the orbit of x.

Define the map $c : C_*(\tilde{X}) \longrightarrow K_G$ by letting

$$c(T) = (g_{T(e_0)}, \ldots, g_{T(e_{\dim T})})$$

This map is trivially equivariant with respect to the action of G.

Definition 7 *The basic chain map is a chain map $c' : K_G \longrightarrow C_*(\tilde{X})$, equivariant with respect to the action of G.*

The construction of the basic map is as follows:

Set $c'(1) = \hat{x}$, where $(1) \in C_0(K_G)$ and $\hat{x}$ is among the fixed representatives of the orbits. Extend the definition to $C_0(K_G)$ by equivariance.

Remark 4 *It is enough to define c' on cells of the form $(1, g_1, \ldots, g_q)$ since no two of these are in the same G-orbit. Then one can extend c' to all of $C_q(K_G)$ by equivariance.*

For $s \in C_1(K_G)$ of the form $(1, a)$, we have $c'(\partial s)$ already defined. But $\pi_0(\tilde{X})$ is generated by some π. Therefore, $c'(\partial s) \pmod{\pi} \in \tilde{H}_0(\tilde{X}) = 0$. So there is a s' so that $c'(\partial s) = \partial s' \pmod{\pi}$. Define $c'(s) = s'$.

In general define $c'(s)$ to be the element s' so that $c'(\partial s) = \partial s'$, which always happens since $H_i(\tilde{X}) = 0$ and $\partial c'(\partial s) = c'(\partial\partial s) = 0$.

Remark 5 *Using a completely analogous construction, we obtain the chain maps Δ, Δ' with the property that*

$$\begin{aligned} \partial\Delta + \Delta\partial &= 1 - cc' \\ \partial\Delta' + \Delta'\partial &= 1 - c'c \end{aligned}$$

Proposition 16 *The chain map c induces an isomorphism $c_e^* : H_e^i(\tilde{X}, R) \xrightarrow{\sim} H_e^i(K_G, R)$.*

Proof: Follows from the fact that the maps c, c' are chain-homotopic to the identity, which implies that c^*, c'^* are also chain-homotopic to the identity. ∎

This proves the previous theorem. ∎

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