Structure of Semisimple Lie Algebras

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Abstract

In this paper I use the adjoint representation of a compact semisimple Lie group to describe a method that completely determines the complex semisimple Lie algebras. The paper has three sections. The first section covers general lemmas and theorems about representations of Lie groups. Proofs for the results presented here can be found in [?] and [?]. The second section determines all possible Dynkin diagrams associated to root systems. The third section relates the Dynkin diagram to the root system, and then the root system to the Lie algebra, to prove that the semisimple Lie algebra is uniquely determined by its Dynkin diagram.

1 Introduction

Let G be a connected compact Lie group and T its maximal torus. We write $\mathfrak{g}$ (resp. $\mathfrak{g}_{\mathbb{R}}$) and $\mathfrak{t}$ (resp. $\mathfrak{t}_{\mathbb{R}}$) for their complexified (resp. real) Lie algebras. Let L be the unit lattice, i.e., the kernel of the exponential map from $\mathfrak{t}$ to T. The weight lattice is $\Lambda = \{\lambda \in i\mathfrak{t}^*_{\mathbb{R}} | e^{\langle \alpha, L\rangle} \subset 2\pi i\mathbb{Z}\}$. In the context of the adjoint representation of G in $\mathfrak{g}$, let Φ be the set of nonzero weights, i.e., the set of *roots*.

By complete reducibility of the adjoint representation we may write

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}^\alpha\right) = \mathfrak{n}_- \oplus \mathfrak{t} \oplus \mathfrak{n}_+$$

where $\mathfrak{g}^\alpha = \{X \in \mathfrak{g} | [X, Y] = \langle \alpha, Y\rangle Y, \forall Y \in \mathfrak{t}_{\mathbb{R}}\}$. Here $\mathfrak{n}_+ = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}^\alpha$ is a Lie subalgebra ($\mathfrak{n}_- = \bar{\mathfrak{n}}_+$).

For $\beta \neq \pm\alpha$ we have that $\{k \in \mathbb{Z} | \beta + k\alpha \in \Phi\}$ is an integer interval whose endpoints add up to $-2\frac{(\alpha,\beta)}{(\alpha,\alpha)}$ (this comes from the properties of finite dimensional irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$. Also, $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = \mathfrak{g}^{\alpha+\beta}$, if the sum is a root, and 0 if it is not a root (or 1-dimensional if the sum is 0).

For any α there is a choice of $E_\alpha \in \mathfrak{g}^\alpha, E_{-\alpha} \in \mathfrak{g}^{-\alpha}, H_\alpha \in i\mathfrak{t}^*_{\mathbb{R}}$ so that $[E_\alpha, E_{-\alpha}] = H_\alpha, [H_\alpha, E_{\pm\alpha}] = \pm 2E_{\pm\alpha}$.

If $W = N_G(T)/T$ is the Weyl group then there exist $s_\alpha \in W$ so that $s_\alpha H_\alpha = -H_\alpha, s_\alpha H = H, \forall (H, H_\alpha) = 0$.

An element $X \in i\mathfrak{t}_{\mathbb{R}}$ is called regular if it is not a zero of any root. Let Φ^+ be the set of positive roots associated with a regular element. A root is called simple if it is in Φ^+ and if it cannot be written as sum of positive roots. The set of simple roots is denoted by Ψ.

The symmetries given by the simple roots generate W. Also, any root β can be written (by construction of Ψ) as

$$\beta = \alpha_1 + \cdots + \alpha_s \tag{1}$$

so that the partial sums $\alpha_1 + \cdots + \alpha_i$ are roots.

For a root α, the only roots that are multiples it are $\pm\alpha$. For $\alpha, \beta \in \Phi$ ($\beta \neq \pm\alpha$) we have that $-2\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$. In this context, if $(\beta, \alpha) < 0$ then $\alpha - \beta$ is a root, and if $(\beta, \alpha) < 0$ then $\alpha + \beta$ is a root. This readily implies that if $\alpha, \beta \in \Psi$ then $(\beta, \alpha) < 0$ or else $\alpha - \beta$ would be a root (using the properties of the set of k so that $\beta + k\alpha$ is a root). This would contradict the fact that α, β are simple.

Moreover $-2\frac{(\beta,\alpha)}{(\beta,\beta)} \in \mathbb{Z}$, and by Cauchy-Schwartz, $(-2\frac{(\alpha,\beta)}{(\alpha,\alpha)})(-2\frac{(\alpha,\beta)}{(\beta,\beta)})$ is at most 3. Therefore $\frac{(\alpha,\beta)}{(\alpha,\alpha)} = 0, -1, -2, -3$.

A root system is irreducible if it cannot be written as the disjoint union of two root subsystems. A root system is irreducible if the simple roots cannot be partitioned into perpendicular sets.

For an irreducible root system, $\frac{(\alpha,\beta)}{(\alpha,\alpha)} = -\frac{1}{4}, -\frac{2}{4}, -\frac{3}{4}$, by swapping α and β if necessary. Therefore the angle between α and β can be $5\pi/6, 3\pi/4, 2\pi/3$. Moreover, the ratios between the lengths of the roots can be $\sqrt{3}$, $\sqrt{2}$ and 1 respectively.

2 Dynkin Diagrams

Consider the space $\mathbb{R}^{|\Psi|}$ with unit vectors $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ representing the simple roots of the root system. The angles between any two vectors is the same as the angle between their corresponding simple roots. From now on I will not distinguish between the vectors and their corresponding simple roots, unless necessary.

The Dynkin diagram associated to a root system is a semi-oriented graph whose vertices are the vectors corresponding to the simple roots (from now on simple roots). There is an arc from a root α to a root β only if the length of simple root α is bigger than the length of simple root β. The number of lines between two roots is 0, 1, 2, 3 if the angle between the roots is $\frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$ respectively.

Note that the Dynkin diagram is connected if an only if the roots cannot be partitioned into two perpendicular sets, if and only if the root system is irreducible. This is a direct consequence of (1), for the procedure in (1) can be reversed.

We will analyze connected Dynkin diagrams. Note that $4(\alpha, \beta)^2$ represents the number of lines between their corresponding vertices in the Dynkin diagram.

Let $d\alpha$ be the degree of α in the Dynkin diagram. Look at the subgraph of the Dynkin graph that merely reduces any nonzero number of edges between two roots to 1. I call this the simplified diagram.

Observations

1. The simplified diagram is a tree.

   Proof: Since we are dealing with simple roots then $\sum \alpha \neq 0$ so $0 < (\sum \alpha, \sum \alpha) = \sum(\alpha, \alpha) + 2\sum_{i<j}(\alpha_i, \alpha_j) = n + 2\sum(\alpha_i, \alpha_j)$. Since $2(\alpha, \beta)$ is 0 if the roots are perpendicular, and ≤ -1 otherwise, it means that there can be at most $n-1$ pairs of roots that are not perpendicular, i.e., that are connected in the simplified diagram. Therefore the simple subgraph is connected and has at most $n-1$ edges so it is a tree. ■

2. $d\alpha \leq 3$ for any α.

   Proof: Assume the contrary. Then let α_1 be connected to $\alpha_2, \dots, \alpha_s$. Since there are no cycles, $(\alpha_i, \alpha_j) = 0$ for $i, j \neq 1$. So if α_1' is the projection of α_1 to the space determined by α_j $(j > 1)$, then $\sum(\alpha_1, \alpha_i)^2 = \sum(\alpha_1', \alpha_i)^2 = ||\alpha_1'||^2 < ||\alpha_1||^2 = 1$ $(\alpha_1 \neq \alpha_1')$. Since $4\sum(\alpha_1, \alpha_i)^2 = d\alpha_1$ it means that $d\alpha_i \leq 3$. Note that $||\alpha_1'|| < ||\alpha_1||$ because α_1 is not in the span of the α_i for $i \neq 1$. ■

3. We call a simple path a sequence $\alpha_1, \dots, \alpha_s$ so that $d\alpha_i = 2$ for $i = 2, \dots, s-1$. No two vertices of degree 3 can be connected by a simple path.

   Proof: Let $\alpha' = \alpha_1 + \cdots + \alpha_s$. Then $(\alpha', \alpha') = s + 2\sum(\alpha_i, \alpha_{i+i}) = s - (s-1) = 1$. Also for α not in the path $(\alpha', \alpha) = (\alpha_1, \alpha) + (\alpha_s, \alpha)$ and at least one of these is zero. Therefore, if we replace the simple path by a vertex α' we get a good graph, but then α' would have degree 4, which cannot be. ■

Observation 3 leaves the following possibilities for Dynkin graphs:

Either all vertices have degree ≤ 2 (which gives the graphs A_n), or exactly one vertex has degree 3, or two vertices have degree 3, and they are linked by a double or triple edge. The triple edge case is easy, for the only possibility is G_2. In the previous case, assume the graph has at least 5 vertices (if it has 4 it has to be F_4).

Let $\alpha_1, \dots, \alpha_5$ be 5 of its vertices so that α_2, α_3 have degree 3. Consider $u = a_1\alpha_1 + a_2\alpha_2, v = a_3\alpha_3 + a_4\alpha_4 + a_5\alpha_5$. From Cauchy-Schwarz we get that $a_2^2a_3^2/2 < (a_1^2 - a_1a_2 + a_2^2)(a_3^2 + a_4^2 + a_5^2 - a_3a_4 - a_4a_5)$. Write $x = a_1/a_2, y = a_4/a_3, z = a_5/a_3$. Then $1/2 \leq (1 - x + x^2)(1 + y^2 + z^2 - y - yz)$. Since we have a choice on a_i, we minimize the RHS. The minimum is $\frac{3}{4}\frac{2}{3} = \frac{1}{2}$ which means we can contradict the inequality. Note that Cauchy-Schwartz must be a strict inequality by linear independence.

We still need to analyze the case when exactly one vertex has degree 3. Either there are 3 simple edges coming out of the vertex, or there is a simple and a double edge coming out of it. The later case is only satisfied by B_n, C_n because one of the vertices that has 2 edges into it must be an endpoint.

For the former case, each of the 3 edges coming out of the vertex determines a subtree. Assume the subtrees have p, q, r vertices respectively. The three subtrees are (by our discussion the subtrees are chains) $(\alpha, a_1, \ldots, a_p)$, $(\alpha, b_1, \ldots, b_q)$ and $(\alpha, c_1, \ldots, c_r)$. Define $a = pa_1 + (p-1)a_2 + \cdots + a_p$, $b = qb_2 + (q-1)b_2 + \cdots + b_q$, $c = rc_2 + (r-1)c_2 + \cdots + c_r$.

We have $||a||^2 = \sum i^2 + 2\sum_{i<j}(p+1-i)(p+1-j)(a_i, a_j) = \sum i^2 - \sum i(i+1) = p(p+1)/2$. Similarly $||b||^2 = q(q+1)/2$, $||c||^2 = r(r+1)/2$. Note that $a/||a||, b/||b||, c/||c||$ are orthogonal, because the subtrees are disconnected in the diagram. Also α is not in their span. Therefore, exactly as we did at observation 2, we get that

$$\begin{aligned} 1 > (\alpha, a)^2 + (\alpha, b)^2 + (\alpha, c)^2 &= \left(\alpha, \frac{pa_1}{\sqrt{p(p+1)/2}}\right)^2 + \left(\alpha, \frac{qb_1}{\sqrt{q(q+1)/2}}\right)^2 + \left(\alpha, \frac{rc_1}{\sqrt{r(r+1)/2}}\right)^2 \\ &= \frac{p}{2(p+1)} + \frac{q}{2(q+1)} + \frac{r}{2(r+1)} \end{aligned}$$

because α is connected to each of a_1, b_1, c_1 by exactly one line and to no other nodes a_i, b_j, c_k. Therefore have that

$$\frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{r+1} > 1$$

The only solutions to this inequality are clearly $(1,2,2), (1,2,3), (1,2,4), (1,1,n)$ and permutations of these. These give the Dynkin diagrams E_6, E_7, E_8, D_{n+3} respectively.

The discussion above yields the following

Theorem 1 *The following are the only possible Dynkin diagrams.*

3 Semisimple Lie Algebras

3.1 Relation Between the Root System and the Dynkin Diagram

Theorem 2 *A connected Dynkin diagram determines the root system up to an isomorphism.*

First I give an algorithm to construct the root system from a connected Dynkin diagram. The diagram gives us a list of all simple roots and relative lengths of them.

Any root is $\beta = k_1\alpha_1 + \cdots + k_n\alpha_n$, where α_i are simple roots (by (1)). We will call $k_1 + \cdots + k_n$ the level of any root. Level 1 roots are simple and they are already known. Level two roots can only be of the form $\alpha + \beta$ with $\alpha \neq \beta$ simple roots, and they will be exactly those sums for which $(\alpha, \beta) < 0$ (see below).

Assume we have all roots of level $\leq N$. For each β of level N, we check whether $(\beta, \alpha) < 0$ for some $\alpha \in \Psi$. This happens if and only if $\beta + \alpha \in \Phi$. Therefore $\beta + \alpha$ obtained this way will cover all roots of level $N + 1$, because any root can be written as in (1).

In order to conclude that the Dynkin diagram uniquely (up to isomorphism) determines the root system, we have to mention that the inner product if also uniquely determined by the Dynkin diagram. This is true because the inner product is bilinear and is uniquely determined on simple roots that generate the root system (by the edges in the Dynkin diagram).

3.2 Relation Between the Lie Algebra and the Root System

The purpose of this section is to discuss the following

Theorem 3 *Two Lie algebras with the same root system are isomorphic.*

First of all, note that any Lie algebra has a system of roots associated to it, i.e., the set of nonzero weights of the adjoint representation.

Let $\mathfrak{g}, \mathfrak{g}'$ be two Lie algebras with the same root systems (for sake of simplicity I replaced 'isomorphic root' systems with 'the same root system').

We have chosen $E_{\pm\alpha} \in \mathfrak{g}^{\pm\alpha}$ and $E'_{\pm\alpha} \in \mathfrak{g}'^{\pm\alpha}$ (such that $E_\alpha = -\bar{E}_{-\alpha}$ and $E'_\alpha = -\bar{E'}_{-\alpha}$). Since $\mathfrak{g}^\alpha, \mathfrak{g}'^\alpha$ are 1-dimensional, the E_α, E'_α generate them. Moreover, a choice of E_α, E'_α for positive roots uniquely determines these elements for the negative roots. Therefore we may look at positive roots for now.

The fact that $\mathfrak{n}_+$ is a subalgebra implies that it is generated by E_α for $\alpha \in \Phi^+$. To be more precise, in the decomposition (1) we would have that

$$E_\beta = [E_{i_s}, [\ldots [E_{i_2}, E_{i_1}] \ldots]. \tag{2}$$

The same thing happens for negative roots. Note that $[E_\alpha, E_{-\alpha}] = H_\alpha$ generates the 1-dimensional space $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$. Therefore, $\mathfrak{g}$ is generated by $E_\alpha, E_{-\alpha}, H_\alpha$.

Therefore the Lie algebra is uniquely determined by the elements $E_\alpha, E_{-\alpha}, H_\alpha$, and hence by E_α. Since our choice is arbitrary for simple roots, the only difference between $\mathfrak{g}$ and $\mathfrak{g}'$ arises from the E_β for non-simple roots β. The only ambiguity in choosing these elements arises from the order of the indexes i_j in (2).

I claim that E_α and E'_α differ only by a constant factor. Since the possibility for several values can occur only because of non-canonical ordering of the simple roots in (1), we should prove the following:

Lemma 4 *The order of simple roots in (1) does affects the values of E_β only by a multiple.*

Proof: (sketch) I will follow the proof given in [**?**]. Let $A = (i_1, i_2, \ldots, i_s)$ and $B = (j_1, j_2, \ldots, j_s)$ be two permutations so that the partial sums of roots with these indexes in (1) are roots. I will write $E_A = [E_{i_s}, [\ldots [E_{i_2}, E_{i_1}] \ldots]]$, and similarly for B. I will show by induction that E_A is a rational multiple of E_B.

For $s = 3$, the Jacobi relation $[E_3, [E_2, E_1]] + [E_2, [E_1, E_3]] + [E_1, [E_3, E_2]] = 0$ means that if $A = (1, 2, 3)$ and $B = (2, 3, 1)$ then $\alpha_1 + \alpha_3$ cannot be a root (if it is, then the two are connected in the Dynkin diagram, so we get a triangle). Therefore, $[E_1, E_3] \in [\mathfrak{g}^{\alpha_1}, \mathfrak{g}^{\alpha_3}] = \mathfrak{g}^{\alpha_1+\alpha_3} = 0$, which means that $E_A = -E_B$.

Now for the induction step, write $\alpha_A = \alpha_{i_1} + \cdots + \alpha_{i_s}$ and same for B. If $i_s = j_s$ then we are done by the inductive hypothesis.

Note that $E_B = q[E_{i_s}, [E_{-i_s}, E_B]]$ (where q is a rational number). This happens because $\alpha_B - \alpha_{i_s} = \alpha_A - \alpha_{i_s}$ is a root. Therefore, the RHS is in $\mathfrak{g}^{\alpha}$, and by dimension 1, it means it has to be a multiple (it is a rational multiple, because E_{-i_s} acts on E_B via inner products that are contained in the Dynkin diagram, and which are rational, since we have the action of E_{i_s} and E_{-i_s}).

Observation: Since $\mathfrak{g}^{\alpha}$ and $\mathfrak{g}^{-\beta}$ are perpendicular for different simple roots, the Jacobi relation tells us that $[E_{-\alpha}, [E_{\beta}, Z]] = [E_{\beta}, [E_{-\alpha}, Z]]$.

Let $j_r = i_s$, $C = (j_1, \ldots, j_{r-1})$. Then as before $[E_{-i_s}, [E_{i_s}, E_C]] = q' E_C$ for some rational a' ($\alpha_C + \alpha_{i_s}$ is a root). Also we have $[E_{-i_s}, E_B] = [E_{j_s}, \ldots [E_{j_{r+1}}, [E_{-i_s}, [E_{i_s}, E_C]]] \ldots]$. This follows from repeated application of the observation (repeated swapping of order).

Now plug in the value of $[E_{-i_s}, [E_{i_s}, E_C]]$ into the previous relation, and get that $E_B = qq'[E_{i_s}, [E_{j_s}, \ldots [E_{j_{r+1}}, E_C] \ldots]]$. Since this has the same number of terms as the original one and has the first index i_s we may apply the inductive hypothesis. ■

Once we have this result, uniqueness follows, because the E_α generate $\mathfrak{g}^{\alpha}$ and E'_α generate $\mathfrak{g}'^{\alpha}$. Since they are linearly dependent, it means that $\mathfrak{g}^{\alpha} \simeq \mathfrak{g}'^{\alpha}$ and then the two Lie algebras will be isomorphic by taking Lie brackets and using the fact that $[\mathfrak{g}^{\alpha}, \mathfrak{g}^{\beta}] = \mathfrak{g}^{\alpha+\beta}$.

One can argue the other way as well. I state the following theorem without proof:

Theorem 5 *For any root system, there is a finite dimensional semisimple Lie algebra.*

A proof of this can be found in [**?**].

According to the stated, but not proved, theorem of existence of Lie algebras for any given root system, each Dynkin diagram corresponds to a unique (up to isomorphism) finite dimensional semisimple Lie algebra. They are the following: $A_n \longrightarrow \mathfrak{sl}(n+1, \mathbb{C})$, $B_n \longrightarrow \mathfrak{so}(2n+1, \mathbb{C})$, $C_n \longrightarrow \mathfrak{sp}(2n, \mathbb{C})$, $D_n \longrightarrow \mathfrak{so}(2n, \mathbb{C})$, and several other special Lie algebras.

References

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